# Objective Estimation for Uncertainty of Restoring Corneal Topography Surface



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## Purpose

Corneal topography (CT) surface can be restored from measurements data by decomposing available data into a set of orthogonal functions (for example, Zernike polynomials) and mapping the CT surface in the desired area. The accuracy of such restoration depends on the number and spacial distribution of available data and on measurement noises. Here we present an algorithm for CT data assimilation, which produces a statistically optimal estimate of measured Zernike amplitudes and their variances. This method yields both CT map and the CT uncertainty map, which is an objective estimation of the restoration accuracy.

## Methods

A statistically optimal estimate of measured Zernike amplitudes can be derived from a combination of the measurement data with a priori information, known before the measurement.

The latter can be obtained from statistics of general population. We performed retrospective analysis of prior clinical data, using available CT data for 312 virgin eyes from to calculate a priori mean and covariance of Zernike coefficients (Figure 1, 2).

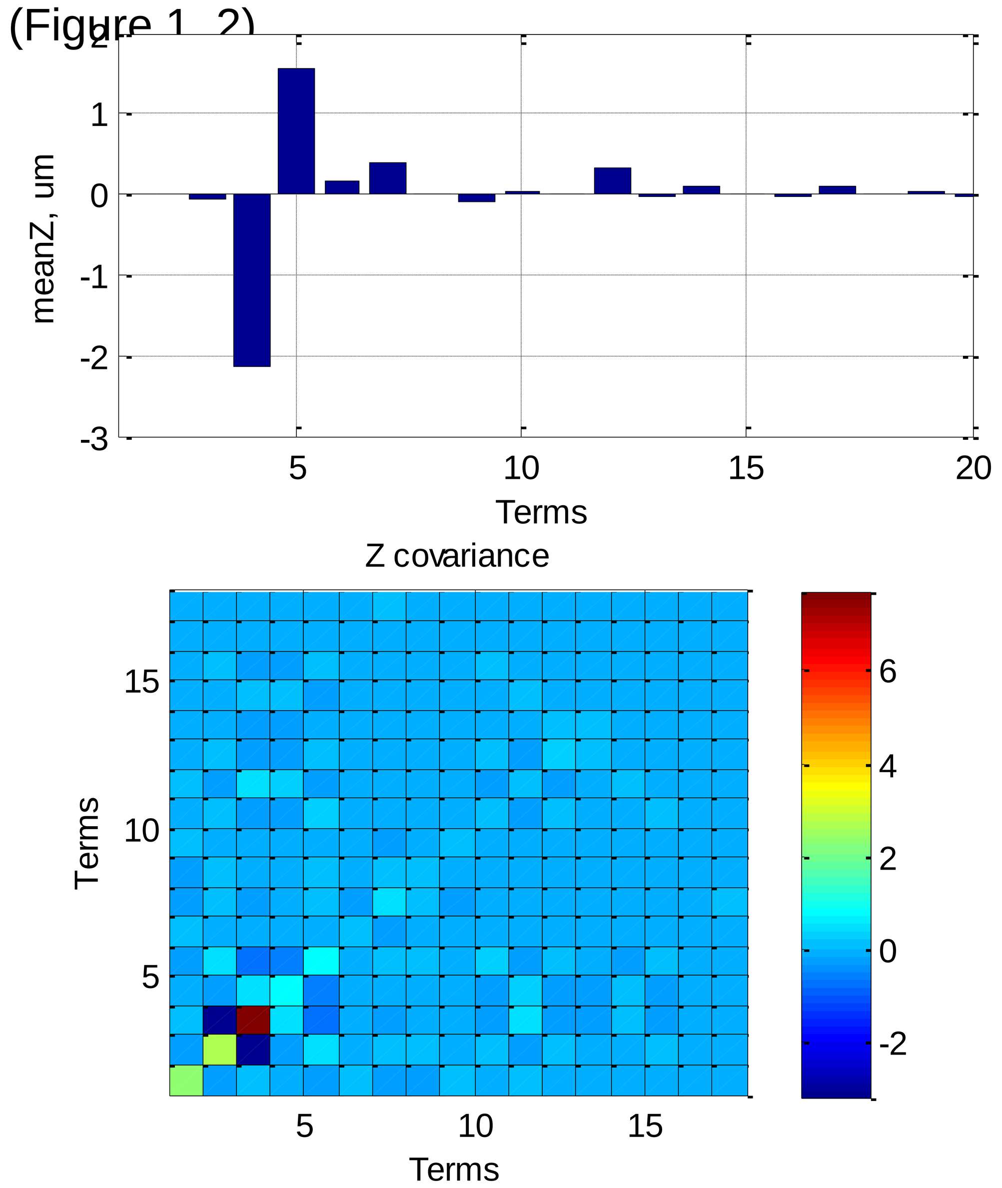


Figure 1. Average and covariance of CT elevation Zernike amplitudes for general population

The Kalman-Bucy technique [1] is used to combine measured CT elevations with a priori mean and covariance of Zernike coefficients. The estimate is computed as follows.

We assume that the system is governed by a linear system:

$$h = \hat{G}A + e \qquad (1)$$

where $h$ is the vector of observations (elevations), $G$ is the coefficients matrix of a linear observation model, and $e$ is the vector of measurement errors. We can use Zernike decomposition of elevation surface as a linear model:

$$h(x,y) = \sum_{i=0}^{N} A_i \cdot Z_i(x,y) \qquad (2)$$

Here $Z$ are Zernike polynomials and $A$ are Zernike amplitudes.

Thus if the elevations field is defined on a set of $M$ points, $(x_k, y_k)$ , the model will be defined by the $MxM$ matrix,

$$G_{i,k} = Z_i(x_k, y_k) \qquad (3)$$

The Kalman-Bucy algorithm, applied to corneal topography measurements, assimilates each measurement by combining the measured data (cornea elevations field) with a priori mean vector, A, and covariance matrix $C_{ik}^{(j)} \equiv \left\langle A_i^{(j)} \cdot A_k^{(j)} \right\rangle$ for Zernike amplitudes, as defined by the following formulas:

$$A_i = A_i^{(prior)} + \hat{K}^{(prior)} \cdot \left[ h_k^{(prior)} - G_{i,k} \cdot A_i^{(prior)} \right]$$
$$\hat{C} = \left\{ \hat{I} - \hat{K}^{(prior)} \cdot \hat{G}^{(prior)} \right\} \cdot \hat{C}^{(prior)} \qquad (4)$$

Here $I$ is the identity matrix having dimensions, $\hat{K} = \hat{C} \cdot G^T \cdot \hat{F}$ is the Kalman-Bucy gain and

$$\hat{F} = \left\{ \hat{G} \cdot \hat{C} \cdot \hat{G}^T + \hat{N} \right\}^{-1}$$

The matrix $N$ is the covariance matrix of measurement errors.

Formulas (4) provide a statistical optimal estimation of the parameter vector, $A$, and its covariance matrix, $C$, after the measurement is assimilated together with prior information.

Once we have the covariance matrix for Zernike amplitudes, we can calculate variance of the corneal elevation field as follows:

$$Var(h(x,y)) = \left\langle \left( \sum_i A_i \cdot Z_i(x,y) \right) \cdot \left( \sum_k A_k \cdot Z_k(x,y) \right) \right\rangle$$
$$= \sum_{i,k} C_{i,k} \cdot Z_i(x,y) \cdot Z_k(x,y)$$

The variance is a measure of CT elevation uncertainty.

## Results

The efficiency of the proposed method is demonstrated using archived corneal topography data from previous clinical studies.

For a given CT measurement the Kalman-Bucy technique assimilates all measured heights on a random set of points, which may contain gaps and low quality values (Fig. 3A). Combining the measurement with a priori information (Fig. 1, 2), the method yields an estimate of Zernike amplitudes and their covariance matrix. The Zernike amplitudes give us the optimal estimate of the measured CT heights while the covariance matrix provides the estimated CT uncertainty (std) in the measured area (Fig. 4 – top row).

These estimations also allow us to reconstruct the entire CT surface and its uncertainty with no gaps in the restoration area (Fig. 4 – bottom row). The estimated uncertainty map gives us a measure of the restoration reliability. The uncertainty is higher at the area edges, because the restoration is based mainly on the data from internal area. Restored field within the measurement gaps, where no or little measurement data are available (compare to Fig. 3) has the highest uncertainty, close to the a priori variance of the general population.

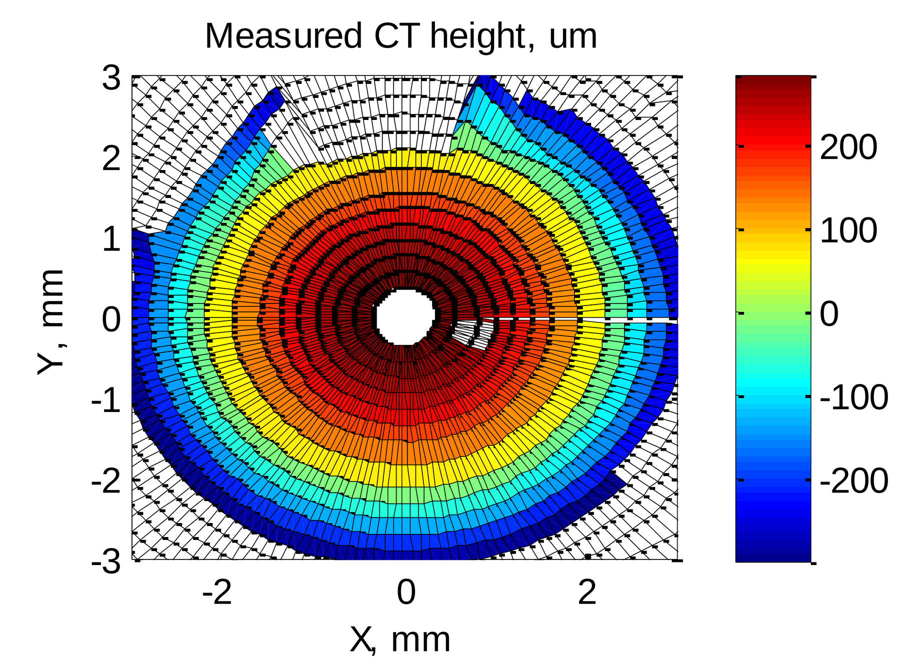


Figure 3. CT height measured with Atlas topographer

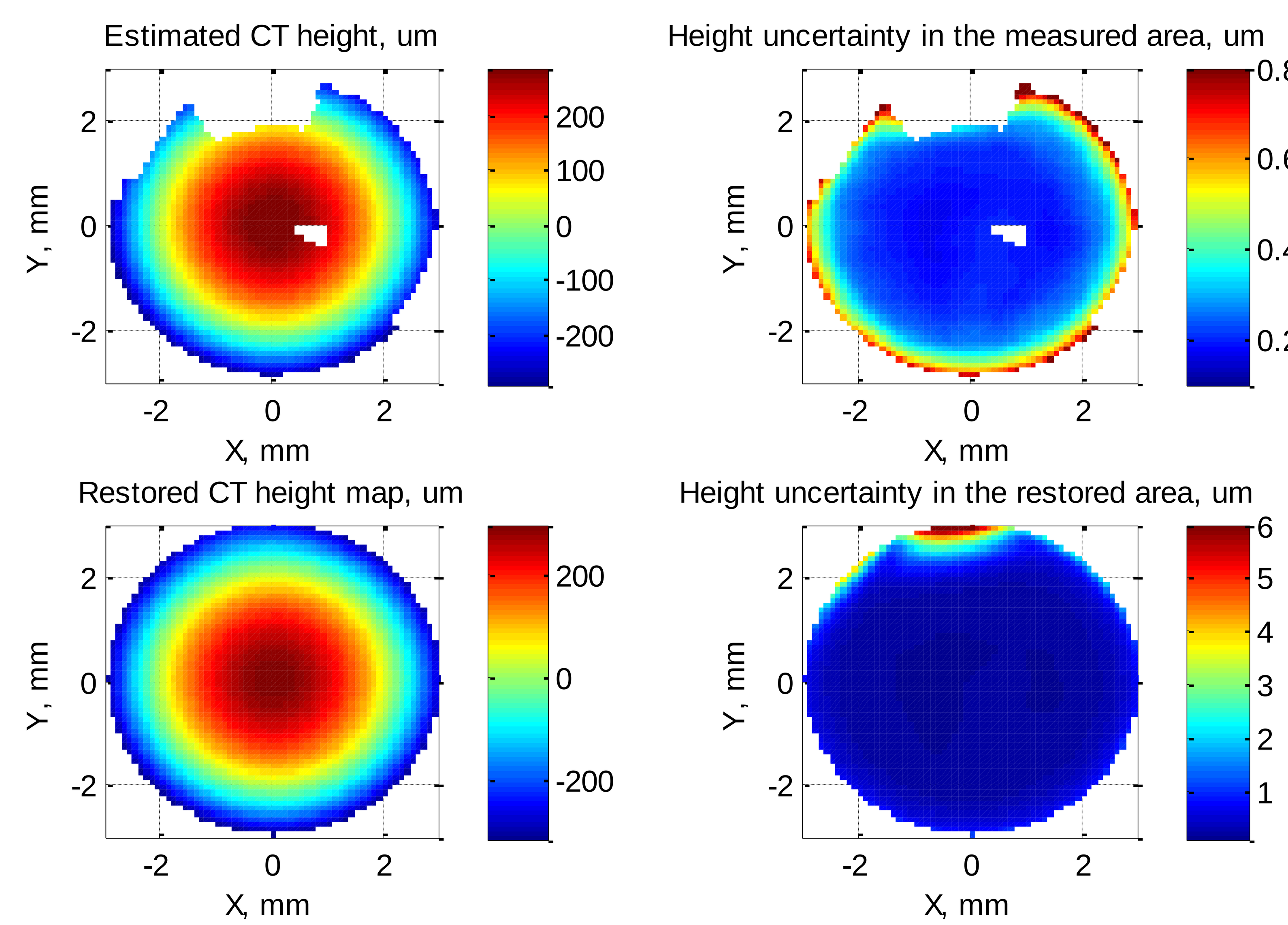


Figure 4. CT surface restoration for CT heights measured with Atlas topographer and restored within 6mm diameter circle. Top row - estimated CT heights and their uncertainty in the measured area. Bottom row – restored heights and uncertainty maps in the entire area.

## Discussion

The Kalman-Bucy algorithm is computationally intensive and needs an efficient numerical implementation. Yet it provides an ultimate data assimilation technique. All available information: both high and low quality measurement points combined together with a priori knowledge of the measured field (general population statistics) are taken into account and weighted by their quality (variance) accordingly. The result of this assimilation is the statistically optimal estimate of the measured field.

Simultaneously, this method yields an objective estimation of the measurement quality. The uncertainty (std) map provides a quantitative reliability measure for the entire measured. It allows quality comparison for different areas of the restored map: higher uncertainty shows up where the measured data are missing or have lower quality.

## Conclusions

The Kalman-Bucy algorithm assimilates measurement data together with a priori information, derived from statistics of general population, which protects the results from measurement outliers. It restores the CT field in the entire area and provides an objective estimate of measurement uncertainty, based on the measurement noise level and the number of available data. The uncertainty map displays the areas where the map is less reliable and to what extent.

## References

1. J.L.Spiesberger, A.L.Fabrikant, A.A. Silivra, H.E. Hurlburt “Mapping Climatic Temperature Changes in the Ocean With Acoustic Tomography: Navigational Requirements”, IEEE Journal of Ocean Engineering, 1997, v.22, n.1 (Appendix B)