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NONLINEAR WAVES IN NONEQUILIBRIUM MEDIA

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INTRODUCTION

By now the most developed – if not to say "chiefly formulated" in the sense of accessibility of analytic results, the development of intuition, and ease of understanding the qualitative aspects of the processes – theory is that of nonlinear waves in transparent equilibrium media where both dissipation and instability are absent. The most important fields of application of this theory are nonlinear optics, nonlinear acoustics, many problems in hydrodynamics and plasma physics, etc. [1-3, 43]. The subject of the present lectures will be the theory of nonlinear waves in nonequilibrium media for which the attenuation and growth of the waves is precisely the most fundamental aspect. The causes of nonequilibrium may be most varied – specifically, uncompensated directional movements, external fields, gradients (of density, of temperature,...), etc. Examples of nonequilibrium media are well known – an electron beam coupled with a low-wave system, a plasma with a multiple-hump velocity distribution function, media with negative conductivity or viscosity – specifically, tunnel and Gunn semiconductors, a boundary layer and other kinds of flow in hydrodynamics, etc.

The abundance of varied instabilities and the absence of conservation laws substantially complicate the problem of formulating a theory of nonlinear waves in nonequilibrium media. The difficulties also increase because in describing such media, which are as a rule essentially nonconservative, it is usually not possible to go over successfully to a uniform description of them on the basis, for example, of the Hamilton or Lagrange formalism. Nevertheless, at least for one-dimensional problems (models), one may hope for the construction of a fairly developed theory if one takes account of the fact that the character of the nonlinear wave processes that occur in nonequilibrium media is determined solely by a finite number of combinations of such parameters as the dispersion, nonlinearity, and the frequency characteristics of the instabilities and of the absorption (i.e., nonconservativeness). It is specifically this fact which allows a unified approach to the description of nonlinear waves in various nonequilibrium media and an attempt to recreate a more or less general picture of wave phenomena in such media on the basis of considering a comparatively small number of fundamental (frequency model) problems.

The principal focus of our lectures will be devoted to an analysis of wave fields for which the inertial interval in k-space (for a spectral description) is a region where there is neither instability nor dissipation and is either small or absent altogether. Under these conditions, we shall distinguish between the following situations (see Fig. 1) – a) the instability is concentrated in the lower portion of the spectrum; b) it is concentrated in the upper portion of the spectrum; c) the intermediate case. Stabilization of the instability may

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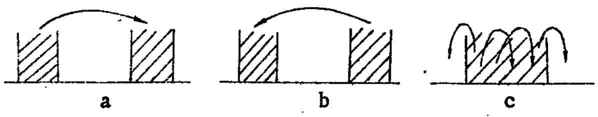


Fig. 1. Possible arrangement of instability and dissipation regions along the spectrum of wave numbers: a) the instability is concentrated in the lower portion of the spectrum, while the dissipation region is in the upper portion. The flow of energy is upward along the spectrum; b) instability is concentrated in the upper part of the spectrum; c) the intermediate case.

be accomplished both by transferring energy to a different region of the spectrum where dissipation predominates and by transmitting it to the same region of the spectrum where linear instability is manifested on the basis of nonlinear losses (viscosity) – Fig. 1c. Let us recall that the source of energy for instabilities may consist of external fields or the main (primary) flow; for envelope waves, such a source may be the carrier field ($\sim \exp(i\omega t)$).

I. TRAVELING ONE-DIMENSIONAL WAVES

Just as for equilibrium media, nonlinear traveling waves in nonequilibrium media can be described successfully in many cases of practical interest within the framework of the so-called one-wave approximation [6] where in view of the smallness of the nonlinearity one may approximately restrict the consideration to wave perturbations of just one form – those belonging to one branch of the dispersion equation (ion-sound waves which travel in one direction in a plasma, traveling capillary waves, rolling waves in a channel, etc.). A formal description of traveling waves can be reduced under these conditions to an analysis of the solutions of several basic equations, some of which are a generalization of the Bürgers and Korteweg–de Vries equations for nonequilibrium media.

1. Waves in Media Having Low-Frequency Instability

Nonlinear waves which develop due to low-frequency instability are most difficult for nonequilibrium media. Their establishment is associated with the transmission (due to nonlinearity) of energy from the instability region upward along the spectrum of wave numbers and the subsequent dissipation of the higher harmonics. Since in the instability region perturbations of various scales may usually grow, it follows that in the presence of a noticeable dispersion in the medium the phases of the individual spectral components are thrown out of kilter and the process is stochasticized – turbulence develops. However, if one of the unstable harmonics is isolated relative to the others (for example, by boundary conditions or a resonant external field), then a dynamic regime of traveling nonlinear waves is established in the medium. Let us add the fact that if even such an isolated specific scale is absent, no randomization of the phases of the individual harmonics takes place in a medium having weak dispersion – groups of harmonics are formed which are slowly correlated with one another, these groups interacting weakly with each other [7-9]. This allows turbulence in the medium having weak dispersion and a low-frequency instability to be also treated as an ensemble of weakly interacting nonlinear waves, each of which is the solution of the dynamic equations of motion of the medium – Bürgers turbulence [10].

a) Media without Dispersion. The Development of Discontinuities. Let us begin with the example of capillary instability of a liquid jet. It is well known [11] that for a cylindrical jet the maximum growth rates correspond to symmetrical perturbations, and it is precisely their development during the nonlinear stage that we will be concerned with. For Reynolds numbers that are smaller than a certain critical value ($Re = \sqrt{\sigma\rho}/\nu_0$; σ is the coefficient of surface tension; ρ is the density; ν_0 is the viscosity), the phase velocity of these perturbations along the jet is equal to zero and does not depend on the wave number – there is no dispersion. The growth rate of the symmetrical perturbation is positive only for small wave numbers (see Fig. 2). With a decrease in Re, the instability boundary k_0 does not change, while the value k^0, at which the growth rate is a maximum, tends to zero. Consequently, for $Re \ll 1$ the development of perturbations in such a system may be treated in the "low-frequency instability" approximation – the growth rate $\gamma(k)$ decays monotonically with increasing k and changes sign for $k = k_0$; for example,

$$\gamma(k) = \gamma - \gamma k^2, \tag{I.1}$$

where γ, ν are determined by the parameters of the jet.

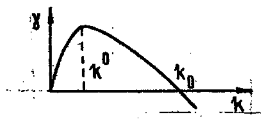


Fig. 2

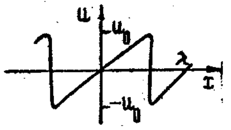


Fig. 3

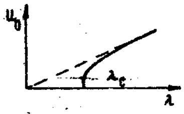


Fig. 4

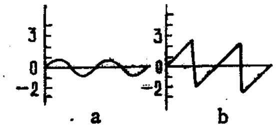


Fig. 5

Fig. 2. The growth rate of axisymmetric capillary oscillations in a cylindrical viscous jet ($Re \ll 1$).

Fig. 3. Shape of all stationary waves in a medium having low-frequency instability ($\lambda \gg \lambda_c$).

Fig. 4. Dependence of the amplitude of stationary waves on wavelength in a medium having a low-frequency instability.

Fig. 5. The result of a numerical calculation of the nonlinear evolution of longwaves in a medium having low-frequency instability ($\nu = 0.1$, $\nu = 1.0$); a) $u(x, 0)$ is the initial condition; b) $u(x, t_0)$ is a steady-state wave.

Having made use of (I.1), we may write the equations for the one-wave approximation for the stream function u [12] as:

$$u_t + uu_x = \gamma u + \nu u_{xx} \tag{I.2}$$

(the velocity components of axisymmetric motion of an incompressible fluid can be expressed in a cylindrical coordinate system in terms of u: $v_\theta = 0$, $v_r = -u_x$, $v_x = u_r$; θ is the azimuthal angle; r is the radial coordinate; x is the coordinate along the jet).

Equation (I.2), which differs from the Bürgers equation by the term γu which is responsible for low-frequency instability, is one of the fundamental (standard) ones in the theory of nonlinear waves in nonequilibrium media. Let us recall that it applies to media without dispersion (more precisely, without reactive dispersion).

Periodic waves excited in a medium described by (I.2) attenuate for $\lambda < \lambda_0 = 2\pi\sqrt{\nu/\gamma}$ and grow to a finite amplitude for $\lambda > \lambda_0$. The amplitude of stationary ($u_t = 0$) waves is difficult to estimate. In the case when their length is only negligibly in excess of the critical length, u may be found by the method of successive approximations and represented in the form $u = \sum_n u_n \sin(knx + \varphi_n)$; then after substitution into (I.2), we find the amplitude of the fundamental harmonic $u_1 = A\sqrt{\lambda - \lambda_0}$ for $(\lambda - \lambda_0) \ll \lambda_0$, where $A = [(12/\pi)\sqrt{\nu\gamma^3}]^{1/2}$ – i.e., stabilization of the instability takes place at the level

$$u \sim \sqrt{\lambda - \lambda_0}. \tag{I.3}$$

The behavior of longwave perturbation $\lambda \gg \lambda_0$ is of greatest interest. As they increase, their profile will be distorted similarly to the profile of a simple wave in an equilibrium medium right up to the formation of steep fronts ("breaks") for which the high-frequency viscosity is already substantial* (such a distortion corresponds to the generation of harmonics – i.e., to the transmission of energy upward along the spectrum). As a result of the dissipation of energy on the front (i.e., in the high-frequency portion of the spectrum), stabilization of the instability takes place – the establishment of a stationary nonlinear wave. In spectral language this process corresponds to establishing a constant energy flux along the spectrum. Since the low-frequency instability has practically no influence on the character of the nonlinear evolution of the perturbation, one may calculate that in the case given, just as in an equilibrium medium, its shape will tend to a sawtooth shape (Fig. 3) having an amplitude†

$$u_0 \approx \gamma\lambda/2; \tag{I.4}$$

if $u_t = 0$, then $u_x = \gamma$ (i.e., $u = \gamma x$ for $-\lambda/2 \le x \le \lambda/2$). Let us emphasize the fact that the amplitude of the steady-state wave does not depend on the magnitude of the attenuation ν which determines only the width of the

*If there were to be no viscosity, then such a "quasisimple" wave would break as it would in a conventional (equilibrium) medium.

† Comparing this with (I.3), we obtain the qualitative dependence of the equilibrium amplitude on wavelength displayed in Fig. 4.

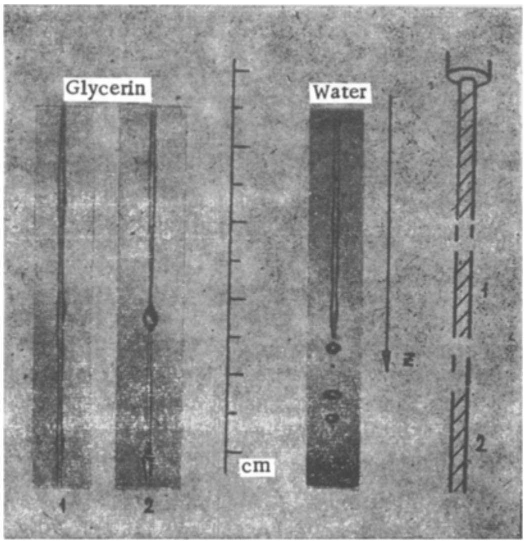


Fig. 6. Evolution of a viscous capillary jet for Re $\ll$ 1 [12].

wave front (i.e., the thickness of the "break"). This thickness Δ may be estimated from the condition of the balance of the processes of nonlinear pumpover of energy upward along the spectrum and high-frequency attenuation $-uu_x \approx \nu u_{xx}$; hence we find

$$\Delta \sim 2\nu/\gamma\lambda \tag{I.5}$$

when (I.4) is taken into account. Figure 5 gives results of a numerical solution of Eq. (I.2) for $u(x, 0) = \sin(\pi x/25) + 0.053\sin(\pi x/50)$, $\nu = 0.1$, $\gamma = 1$. It is evident that in the process of evolution of an almost harmonic perturbation, a sawtooth wave develops which turned out to be unstable relative to longwave perturbations that lead to the establishment of a wave having the maximum possible period.* In the actual situation, this period may be determined either by the initial or the boundary condition.

Returning to the problem of axisymmetric oscillations of viscous jets, we note that the sawtooth wave shape of the stream function corresponds to a solitonlike wave having a transverse velocity $v_r = \partial u/\partial x$. Therefore, as a result of the development of the instability, the jet takes the form of a thin filament with bead drops deposited on it. It is precisely such a picture which is observed experimentally (see Fig. 6), but only during the intermediate stage of evolution. The point is that the filament, in turn, is unstable relative to the development of new small satellite drops between beads. This secondary instability [it has likewise been described by (I.2)] has a growth rate that is much smaller than the original instability. In view of the bounding of a filament section by the large drops, the scale of the growing secondary perturbations will be smaller than that of the primary perturbation; i.e., the satellite will be considerably smaller than the drops which are formed immediately (see Fig. 6). Third-order instability leading to the development of still smaller satellites, etc., may be treated analogously right up to disruption of the conditions governing applicability of Eq. (I.2).

b) The Structure of a Discontinuity. The Effect of Dispersion. It is easy to confirm the fact that in the absence of dispersion [the Bürgers equation is applicable for a nonequilibrium medium (I.2)] the wave front will be smooth. Actually, by considering the stationary solutions $u = u(\xi = x - Vt)$, $V = \text{const}$ of Eq. (I.2), we find the following equation for $V = 0$ as a result of integration:

$$\frac{u^2 - u_0^2}{2} = \nu\left[u_\xi - \gamma \ln\frac{\gamma - u_\xi}{\gamma}\right], \tag{I.6}$$

where u_0 is a parameter characterizing the amplitude. From (I.6) it follows that the bounded stationary solutions are necessarily periodic, their shape approaching the sawtooth shape with a growth of u_0 [the phase diagram plotted on the basis of (I.6) is displayed in Fig. 7]. Waves having a large amplitude have a segment of

*This result can easily be understood if one remembers that greater periods correspond to a greater increment.

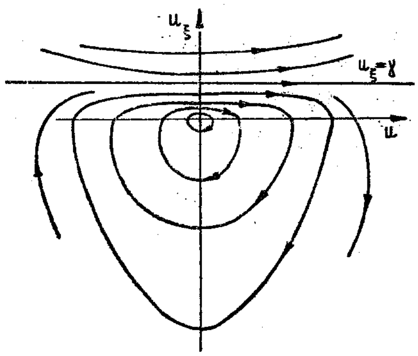


Fig. 7. Phase diagram of the stationary solutions of Eq. (I.2). All of the closed curves are situated below the straight line $u_\xi = \gamma$.

slow variation – on the phase plane this segment corresponds to motion near the straight line $u_\xi = \gamma$; they also have a segment with rapid variation – motion along a loop that departs far downward.*

In order to answer the question as to how dispersion affects the structure of a discontinuity, one should modify Eq. (I.2) by introducing a term responsible for dispersion into it. Let us do this using the example of ion-sound waves in a nonequilibrium plasma.

As is well known [13], in a nonisothermal plasma ($T_e \gg T_i$) it follows that as a result of the relative motion of electrons and ions for

$$v_{Te}\sqrt{\frac{m}{M}} = v_s < V \ll v_{Te} \tag{I.7}$$

(v_s is the velocity of ion sound; V is the velocity of mutual drift; v_{Te} is the thermal velocity of the electrons) it is possible for instability relative to longwave ion-sound perturbations to occur. The character of this instability is qualitatively different for small and large numbers of collisions. Here let us consider the limiting case of a high collision frequency when the use of the hydrodynamics approximation is possible.† The original equations are those of the two-fluid model [14]:

$$\begin{gathered}
\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0, \\
\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e u_e) = 0, \\
\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} = -\frac{\partial \varphi}{\partial x} + \nu_i \frac{\partial^2 u_i}{\partial x^2}, \\
\frac{\partial \varphi}{\partial x} - \frac{1}{n_e}\frac{\partial}{\partial x}(n_e T_e) = 0, \\
\frac{\partial T_e}{\partial t} + n_e \frac{\partial T_e}{\partial x} + \frac{2}{3} T_e \frac{\partial u_e}{\partial x} = \frac{2}{3}\varkappa \frac{\partial^2 T_e}{\partial x^2}, \\
\frac{\lambda_D^2}{l^2}\frac{\partial^2 \varphi}{\partial^2 \varphi} = n_e - n_i,
\end{gathered} \tag{I.8}$$

where λ_D is the Debye radius; l is the characteristic perturbation length; $n_{e,i}$ are the electron and ion concentrations; φ is the field potential; T_e is the electron temperature; ν_i is the viscosity; $\varkappa$ is the electron thermal conductivity.‡ Here it follows that in the Navier–Stokes equation for the ions the ion pressure is neglected (in view of the smallness of T_i), while in the equations for the electrons the electron inertia is neglected [i.e., the small term $(m/M)(du_e/dt)$ is neglected]. The continuity, thermal conductivity, and Poisson equations are written in conventional form.

In the model considered there are two dissipation mechanisms – ion viscosity and electron thermal conductivity which is assumed fairly large here. If $V > v_s$, then the electron thermal conductivity does not

*Bounded solutions in the class of stationary waves exist in (I.2) only for V = 0, it being true that, independently of the period, all perturbations propagate with an identical velocity – the phase velocity of small perturbations. It is natural that such a situation is possible only in a medium having no dispersion.

†The case of a collisionless plasma is considered below – see Para. 2b.

‡In the model considered, electron–ion collisions are not taken into account.

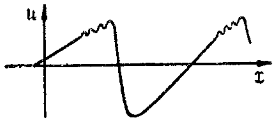


Fig. 8

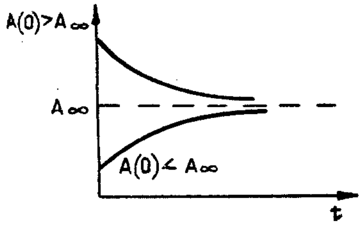


Fig. 9

Fig. 8. The effect of weak dispersion on the shape of the stationary wave in a medium having a low-frequency instability [14] ($\beta > \nu^2/2\gamma\lambda$).

Fig. 9. Dynamics of solitons in a medium having low-frequency instability. The amplitude of the solitons tends to the constant A_∞ for $t \to \infty$.

lead to wave damping but to low-frequency instability whose development and subsequent stabilization due to the ion viscosity may be described in the one-wave approximation by the equation [14]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = \gamma u + \nu\frac{\partial^2 u}{\partial x^2}, \tag{I.9}$$

where $\gamma = (V/v_s - 1)/2\varkappa$; $\beta = \lambda_D^2/l^2$; $\nu = \nu_i/2$. Unlike (I.2), in this equation, which may be considered to be generalized Korteweg–de Vries–Bürgers equations for nonequilibrium media, the high-frequency dispersion is taken into account: $\beta \neq 0$.

In an equilibrium medium ($\gamma = 0$) for a fairly strong dispersion $\beta > \nu^2/2\Delta u$ (Δu is the amplitude of the discontinuity), attenuating oscillations appear on the wave front. Since in the case considered the nonequilibrium determines only the amplitude of the steady-state wave, it is natural to expect that the same result will also be obtained for $\gamma > 0$. Figure 8 displays the result of a numerical calculation for (I.9) for $\beta > \nu^2/2\gamma\nu$, which confirms our intuition.

c) Solitons in Media Having Low-Frequency Instability. If the dispersion is so great that the oscillations cut up the entire wave profile, then it is natural to speak of the behavior of the oscillations proper rather than of the front and its structure (i.e., to speak of the solitons). It turns out this way for

$$\beta u''' \sim uu' \gg \gamma u \sim \nu u''. \tag{I.10}$$

Keeping in mind the fact that for small γ and ν the parameters of solitons vary slowly, it is convenient to make use of the method of averaging over stationary waves in analyzing a nonstationary process [6, 15]. The solution is represented in the form of a soliton having an amplitude, width, and velocity that varies slowly:

$$u = A(t)\,\mathrm{ch}^{-2}\left[\left(x - \int_0^t V(t)\,dt\right)/\lambda(t)\right], \tag{I.11}$$

where $V(t) = \frac{1}{3}A(t)$, while $\lambda^2(t) = 12\beta/A(t)$ (i.e., the parameters of the soliton are related to one another in the same way as they are in a transparent medium described by the conventional KDV equation). After substitution of (I.11) into the original equation (I.9) written in the form of the conservation law:

$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{u^2}{2}dx + \int_{-\infty}^{\infty} uM[u]\,dx = 0, \tag{I.12}$$

where $M[u] = -\gamma u - \nu u$ is the dissipative operator, the equation for the amplitude is obtained [14]:

$$\frac{dA}{dt} = \frac{4}{2}\gamma A - \frac{4}{45}\frac{\nu}{\beta}A^2. \tag{I.13}$$

From its solution

$$A(t) = \frac{A_0 A_\infty e^{\alpha t}}{A_0(e^{\alpha t} - 1) + A_\infty} \tag{I.14}$$

(A_0 is the initial amplitude; $A_\infty = 5\gamma/\nu$; $\alpha = 4\gamma/3$) it follows that solitons having an initial amplitude $A_0 < A_\infty$ are accelerated to the velocity $V_\infty = \frac{1}{3}A_\infty$ and amplified to A_∞, while solitons having $A_0 > A_\infty$ are decelerated and attenuate to A_∞ (see Fig. 9). As a result, for $t \to \infty$ an operating mode is established in the form of solitons having identical amplitudes and velocities. Physically, this result seems to be fairly obvious – solitons of low amplitude have a large width, and low-frequency instability leads to their amplification; narrow solitons (having a large amplitude) attenuate due to high-frequency.

2. Waves in Media Having Negative Viscosity

It is clear that the actual medium cannot be of a nonequilibrium nature with respect to perturbations having arbitrary space – time scales. Moreover, in the majority of cases the nonequilibrium condition is manifested only for a comparatively narrow "strip" of perturbations (i.e., clearly delineated scales exist). For example, for hydrodynamic instability of flows [16, 17] the characteristic dimension of the stream is such a scale, while for beam instability in a plasma the reciprocal plasma frequency serves as the time scale, etc. The most typical mechanism for stabilization of instability in these cases, as has already been said, resides in the transfer of energy upward along the spectrum where it dissipates – for example, due to viscosity.

However, comparatively recently it became clear that another kind of nonequilibrium media exists for which growth rates that increase with frequency ("negative viscosity") are manifested when waves are propagated in them. The situation is specifically thus, for example, for propagation of waves in a medium with a turbulence characterized by a short correlation time. Under these conditions, the wave increases due to the transfer of energy from small scale to large scale (a process that is the reverse of the Richardson cascade transfer of energy from "larger vortices to smaller vortices" [18]). The negative-viscosity mechanism is evidently realized for currents in the ocean and in the atmosphere [19].

a) Explosive Instability of Traveling Waves. Waves on a Draining Film. The effect of negative viscosity is manifested in completely familiar and clear (at first glance) phenomena. As an example, let us take a layer of viscous fluid having a thickness H (see Fig. 10) which is draining along an inclined plane having an angle of inclination α. Plane-parallel stationary flow has a parabolic velocity profile U(y):

$$U(y) = \frac{g \sin \alpha}{2\nu}(2Hy - y^2), \tag{I.15}$$

where y is the coordinate normal to the surface of the flow; U(y) is the velocity component parallel to the inclined plane; ν is the kinematic viscosity, while g is the free-fall acceleration.

The investigation of the linear stability of such a current leads to the results displayed graphically in Fig. 11. Here a neutral curve – the boundary between the stability and instability domains – is drawn on the k, Re plane [k is the wave number of the harmonic perturbations; $Re = (g \sin\alpha H^3)/2\nu^2$ is the Reynolds number]. This boundary consists of two branches. The upper branch is a slightly deformed neutral curve for Poiseuille flow (between two planes) with a parabolic velocity profile [16]. For it there exists a critical Reynolds number R_* which is such that when it is exceeded ($Re > Re_*$) results in a situation in which Tollmien – Schlichting waves [16, 20] grow in the flow; this may then lead to the development of strong turbulence.

The lower branch of the neutral curve likewise has a critical Reynolds number

$$Re_{cr} = \frac{5}{4} \operatorname{ctg} \alpha; \tag{I.16}$$

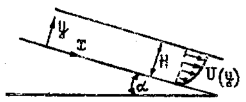


Fig. 10

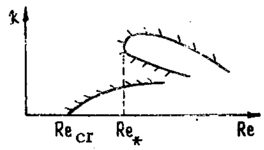


Fig. 11

Fig. 10. Laminar drainage of a viscous fluid downward along an inclined plane.

Fig. 11. Neutral curves for flow along an inclined plane. The hatching is directed toward the stable domain.

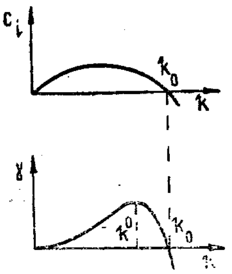


Fig. 12

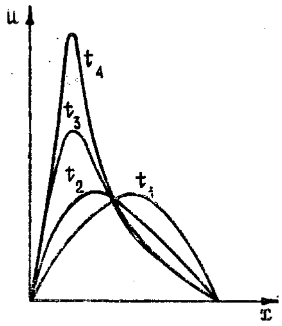


Fig. 13

Fig. 12. Dependence of the imaginary part of the phase velocity $c_i(k)$ and of the growth rate $\gamma(k) = kc_i(k)$ on the wave number for perturbations of flow along an inclined plane for Re = const.

Fig. 13. Evolution of a nonlinear wave in a medium having negative viscosity. The shape of the wave is depicted for the successive times $t_1 < t_2 < t_3 < t_4$ [24].

when this value is exceeded, surface gravitational-capillary waves are excited (the flow along the vertical plane is evidently always unstable: $Re_{cr} = 0$).†

For the flows that are usually observed under laboratory conditions (water or alcohol; H ~ 0.01-1 cm), the condition $Re_* \gg Re_{cr}$ is obtained (i.e., the flow may always be considered laminar, and the investigation may be restricted solely to instability of surface waves). For a stipulated Re, the growth rate $\gamma(k)$ of this instability rises with the growth of the wave number k, and for a certain $k = k^0$ is a maximum (see Fig. 12). For perturbations having $k \ll k^0$, which corresponds to $\gamma(k) \sim k^2$, the equation in the one-wave approximation which describes this contains a term which is responsible for negative viscosity [21, 22]:

$$\dot{\eta} + v\,\eta_x + d\,\eta\eta_x + \beta\eta_{xxx} + \nu\eta_{xx} = 0, \tag{I.17}$$

where η is the deviation of the surface level; v, d, β are constants which are determined by the parameters of the flow and by the surface tension, while $\nu \sim (Re - R_{cr})$ is the effective viscosity which becomes negative for $Re > Re_{cr}$. This same equation also describes long waves which travel along an inclined channel having an arbitrary cross section [22].

In the new variables, $-x' = x - vt$, $u = d\eta$, Eq. (I.17) has the form (for $\beta = 0$)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{I.18}$$

i.e., it differs from the Bürgers equation solely in the sign of ν and goes over into that equation with the substitution $t \to \partial - t$, $x \to -x$, $u \to -u$. This equation permits an exact solution to be obtained: As a result of the substitution $u = (2\nu/\theta)(\partial\theta/\partial x)$, it can be reduced to the linear thermal-conductivity equation

$$\frac{\partial \theta}{\partial t} + \nu\frac{\partial^2\theta}{\partial x^2} = 0. \tag{I.19}$$

It can easily be demonstrated [24] that the smooth solution of (I.18) at first evolves as a simple wave – its profile wraps around and then goes to infinity in a finite time (see Fig. 13): This is explosion instability of a traveling wave. This result may be clarified in spectral language as follows. An energy flux from the low-frequency modes which increases as the high-frequency harmonics are amplified is added to the exponential growth of the high-frequency harmonic; it is this which leads to the nonlinear growth rate of the high-frequency modes which ensure the explosion.

† The condition for instability of surface waves is sometimes written in the form $\Phi > \sqrt{5/2}$, where $\Phi = U_{max}/\sqrt{gH\cot\alpha}$ is the Froude number.

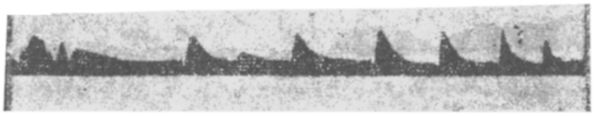


Fig. 14. Shape of long waves on a film that drains along a large-diameter vertical pipe [25].

It is natural that in an actual situation the explosion stage is not reached: either nonlinear attenuation is manifested, or else absorption at higher frequency. In our case – long waves on the surface of a draining liquid – it is necessary to take account of the attenuation in the region of small scale $\lambda < 2\pi/k^0$ (see Fig. 12). Steady-state (as a result of the limitation of the explosive instability) long nonlinear waves on the surface of a vertically draining liquid film are displayed in Fig. 14 [25]. For shorter waves, it is already the dispersion [$\beta \neq 0$ in (I.17)] that is essential, and oscillations appear on the profile of the wave (see Fig. 15). Waves whose length is close to the critical wavelength $\lambda_0 = 2\pi/k_0$ will evidently be quasisinusoidal (Fig. 15a) [26].

b) Ion-Sound Waves in a Collisionless Plasma. Let us return to the example of ion-sound waves in a nonequilibrium plasma which was considered above. We spoke of the case in which the ion viscosity was great and derived the equation for the one-wave approximation (I.9) for nonlinear waves, where the instability is described by the term γu. Under these conditions, the growth rate $\sim(\gamma - \nu k^2)$. In the other limiting case – a collisionless plasma (no viscosity), the growth rate of ion-sound waves, which is determined by resonance interaction with electrons with ions, is already different:

$$\gamma = \sqrt{\frac{\pi m}{8M}}(V - v_s)\,|\,k\,| \tag{I.20}$$

– it is proportional to the modulus of the wave number. Such an instability can no longer be taken into account within the framework of partial differential equations, and the "standard" (model) equations for the one-wave approximation becomes integral* [28] (n is the density):

$$\frac{\partial n}{\partial t} + \alpha n \frac{\partial n}{\partial x} + \beta \frac{\partial^3 n}{\partial x^3} = \gamma \int_{-\infty}^{\infty} \frac{\partial n}{\partial \xi} \frac{d\xi}{x - \xi}. \tag{I.21}$$

In an equilibrium plasma we have V = 0 and the integral term ($\gamma < 0$) describes resonance attenuation of ion sound on electrons. In the presence of current, when $V > v_S$, the attenuation is replaced by instability. For small scales, Landau damping on ions becomes substantial [27], and therefore the next term in the expansion

*An analogous description is given for nonlinear waves which grow as a result of resonant interaction of waves with particles or of waves with fluxes, and in other cases such as wind waves on shallow waters, Langmuir waves in a plasma with a beam [29], etc. The same integral term should also be added in Eq. (I.2) for the stream function in a capillary jet. It takes account of a deviation $\sim|k|$ of the real growth rate from the approximation (I.2).

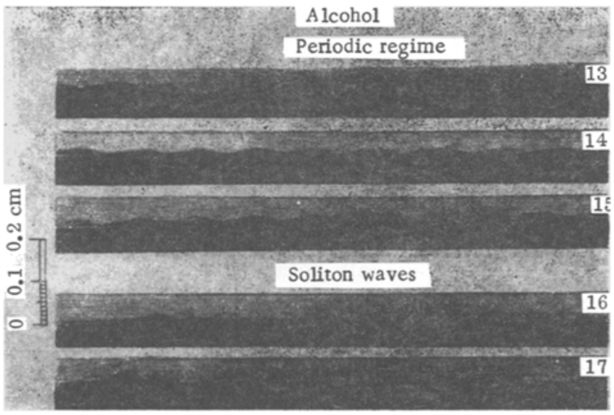


Fig. 15. Waves of various lengths on a draining film [26]. The shape of the wave varies with wavelength – from a quasisinusoid to a relaxation wave.

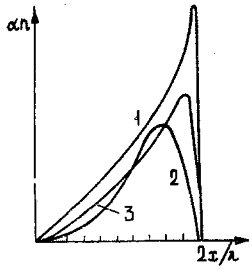


Fig. 16. Shape of the stationary ion-sound waves in a collisionless non-equilibrium plasma: 1) $\nu = 0.5$; 2) $\nu = 1$; 3) $\nu = 2$.

of the total growth rate in k yields a term in (I.21) which is analogous to the viscosity:

$$\frac{\partial n}{\partial t} + \alpha n \frac{\partial n}{\partial x} + \beta \frac{\partial^3 n}{\partial x^3} = \gamma \int_{-\infty}^{\infty} \frac{\partial n}{\partial \xi} \frac{d\xi}{x - \xi} + \nu \frac{\partial^2 n}{\partial x^2}. \tag{I.22}$$

As a result of stabilization of the instability, the nonlinear stationary waves reach a steady state, and the shape of these waves can be investigated successfully only by numerical methods. Figure 16 shows the dependence of the shape of these waves on various ν for $\beta = 0$ [14]. It is obvious that with a growth of the wavelength (or, what amounts to the same thing, a reduction of the damping) its leading edge become steeper. In accordance with the results of this calculation, the average energy of the nonlinear wave increases with a growth of wavelength considerably more slowly than it does in the case of low-frequency instability [compare with (I.3)]:

$$\langle n^2 \rangle \sim \sqrt[4]{\lambda}. \tag{I.23}$$

Now it is clear that in the case when the growth rate $\gamma|k|$ is added to the low-frequency growth rate: $\gamma(k) = \gamma_1 + \gamma_2|k| - \nu k^2$ [as in the more rigorous analysis of nonlinear waves on a cylindrical jet – see footnote to (I.21)], the profile of the stationary waves near the front must become steeper. Specifically, for a capillary jet this result means the appearance of constrictions on both sides of each drop being formed [12]. Such constrictions actually are observed (see Fig. 6).

II. INTERACTION OF STABLE AND UNSTABLE WAVES

In equilibrium media waves may exchange energy only with one another, and for their dynamic interaction a periodic process arises. In the simplest case of three waves, the high-frequency wave gives up its energy to the low-frequency wave during decay, and then receives this energy back during merging of the waves. However, if the phases of the waves are random, then a uniform distribution of energy among the modes is established. In nonequilibrium media, the waves may also take energy from the medium, and the wave interactions turn out to be considerably more varied and abundant. For example, under these conditions it is already possible for a low-frequency wave to decay; as a result of interactions it is possible for simultaneous growth of all of the waves to occur – explosion instability, the phases of the waves are not necessarily randomized for a large number of interactions, and they may even be synchronized due to the growth of individual modes, etc.

In this and the subsequent sections we examine mainly three-wave interactions in media having a quadratic nonlinearity. We shall discuss the interaction of Tollmien–Schlichting waves in a boundary layer, waves in an electron beam–plasma system, in semiconductors with a nonlinear dependence of the current density on the field, and in certain other media.

1. Averaged Equations

First of all, let us consider the formal aspects of three-wave interactions. In media having a low quadratic nonlinearity, the interaction of resonance-coupled waves ω_1, $\mathbf{k}_1$; ω_2, $\mathbf{k}_2$; ω_3, $\mathbf{k}_3$, where $\omega_3 = \omega_2 + \omega_1 + \Delta\omega$, $\mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_1$, may be described by means of the averaged equations for the complex amplitudes of the waves:

$$\begin{aligned} \frac{\partial a_{1,2}}{\partial t} + v_{1,2} \nabla a_{1,2} &= \sigma_{1,2}\, a_3\, a^*_{2,1}\, e^{i\Delta\omega t} + \nu_{1,2}\, a_{1,2}, \\ \frac{\partial a_3}{\partial t} + v_3 \nabla a_3 &= \sigma_3\, a_1\, a_2\, e^{-i\Delta\omega t} + \nu_3\, a_3. \end{aligned} \tag{II.1}$$

The nonuniformity of the medium may be manifested in these equations in two ways. On the one hand, the interaction coefficient σ, which in equilibrium media always satisfy symmetry relationships of the type

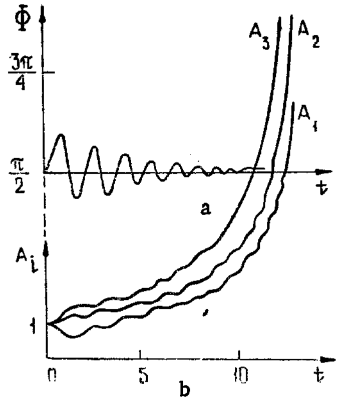


Fig. 17. Synchronization of the phases of the waves and the departure of their amplitudes to infinity during explosion in stability.

$\sigma_1 = \sigma_2 = -\sigma_3$ (for $\sigma = \sigma^*$),† may turn out in a nonequilibrium medium to be arbitrary and complex, resulting in the appearance of "nonlinear" instabilities. On the other hand, σ may remain the same as they are in the equilibrium case, but the individual waves will turn out to be unstable even in the linear approximation $\nu_i > 0$. Naturally, such "pure" situations do not exhaust all possibilities – combined cases are also encountered; however, one may frequently neglect the small linear growth rates in comparison with nonlinear growth rates or small imaginary corrections to σ.

Let $\nabla = \Delta\omega = \nu = 0$. Using the substitution $a_j = (A_j/\sqrt{|\sigma_i\sigma_k|})e^{i\varphi_j}$ ($k \neq i \neq j$), i, j, k = 1, 2, 3, we write (II.1) in real form:

$$\dot{A}_1 = A_3 A_2 \cos(\Phi + \theta_1), \quad \dot{A}_2 = A_3 A_1 \cos(\Phi + \theta_2),$$
$$\dot{A}_3 = A_1 A_2 \cos(\Phi + \theta_3), \tag{II.2}$$
$$\dot{\Phi} = -\frac{A_1 A_2}{A_3}\sin(\Phi + \theta_3) - \frac{A_2 A_3}{A_1}\sin(\Phi + \theta_1) - \frac{A_1 A_3}{A_2}\sin(\Phi + \theta_2).$$

The interacting wave demonstrates a qualitatively different behavior for different values of θ_i. Periodic exchange of energy (an equilibrium medium) corresponds to the case $\theta_1 = \theta_2 = \theta_3 \pm \pi$. For explosion instability it is necessary and sufficient that the phase difference Φ change only in such limits that all $\cos(\Phi + \theta_{1,2,3})$ are sign-positive. This case is realized when all θ_i have a relationship such that $\max\{\theta_i\} - \min\{\theta_i\} < \pi$ (i.e., lie on one half-plane [30]). Under these conditions, the phases of the waves are rapidly synchronized and their amplitudes go to infinity in a finite time (see Fig. 17).

Assuming now that σ are the same as they are in an equilibrium medium, let us give detailed consideration to the interaction of waves that are stable and unstable in the linear approximation.

2. Amplification of Weak Signals

Let the field of one of the waves be stipulated. If the medium were to be an equilibrium medium, the situation in which the amplitudes of the low-frequency waves are small would be of greatest interest here (i.e., parametric amplification of these waves as a result of the high-frequency pumping ω_3 (decay instability). Consideration of the nonequilibrium nature of the medium in the linear approximation – the appearance of a linear growth rate for one of the waves – may lead to qualitatively new effects only in the case when the parametric growth rate turns out to be significantly below the growth rate of the unstable wave. As we shall now verify, under these conditions the transfer of the large growth rate of the unstable wave to the weak signal wave takes place, in view of which such a process is sometimes called superheterodyne amplification [31]. This process is observed in physics and computer experiments on the interactions of stable or nonincreasing and unstable waves in piezosemiconductors [31, 32], in nonequilibrium plasma [33], etc. It is essential that such an "amplifier" will also operate for low-frequency pumping [33] (which is impossible in an equilibrium medium).

Let us consider parametric amplification in a one-dimensional medium (along x) for a signal wave $a_3(x)$ in the stipulated field of a pumping wave $a_2 = a_{20} = \text{const}$ on the assumption that $\partial/\partial t = \Delta\omega = \nu_{2,3} = 0$; i.e., instead of (II.1) we have

† It is not difficult to verify the fact that in an equilibrium medium one may always choose the variables in such a way that σ will be either purely real and this condition will be satisfied, or purely imaginary in which case the symmetry conditions are written as $\sigma_1 = \sigma_2 = \sigma_3$.

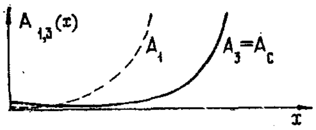


Fig. 18. Abrupt growth of the signal wave far from the boundary, which is due to the effect of superheterodyne amplification.

$$\frac{\partial a_1}{\partial x} = \sigma_1 a_3 a_2^* + \gamma a_1,$$
$$\frac{\partial a_3}{\partial x} = -\sigma_3 a_1 a_2, \quad \sigma_i = \sigma_i^*, \quad \omega_3 = \omega_1 + \omega_2. \tag{II.3}$$

If the parametric coupling is weak (i.e., the ratio of the parametric growth rate $\Gamma = A_{20}\sqrt{\sigma_1\sigma_3}$ to γ is $\delta = (\Gamma/\gamma)^2 \ll 1$, then for the boundary conditions $a_3(x = 0) = a_3(0)$, $a_1(x = 0) = 0$ we have the following approximate relationship for (II.3):

$$a_3(x) = a_3(0)(1 - \delta e^{\gamma x}),$$
$$a_1(x) = a_3(0)\frac{\sigma_1 a_{20}^*}{\gamma}(e^{\gamma x} - 1). \tag{II.4}$$

Based on this solution, it is easy to trace the individual stages of the superheterodyne amplification for propagation of the wave: 1) amplification of the free wave a_1, which takes account of the information on the signal due to the boundary conditions (this process occupies the interval $0 < x \lessapprox 1/\gamma$; 2) strong growth of the free ("heterodyne") wave a_1 – this stage occupies the interval $1/\gamma \lessapprox x \lessapprox x_0 = (1/\gamma)\ln(1/\delta)$ and is completed by the transfer of the large growth rate γ to the signal wave; 3) amplification of the wave a_3 with the growth rate γ by a factor $k \gg 1$ over the distance $x = (1/\gamma)\ln k/\delta$ [for $x < x_0$ this wave is practically unamplified and is equal to $a_3(0)$]. Thus, the experimental result consisting of the abrupt growth of the weak wave having a small parametric growth rate far from the boundary of the nonequilibrium medium which appears to be paradoxical at first glance may have a trivial explanation – initially the hidden process of the transfer of the large growth rate of the other wave to this wave goes on, and then abrupt growth occurs which essentially bears no relationship to parametric amplification (see Fig. 18 [33]).

In a real nonequilibrium medium, the considered process is limited by two factors – the growth of the fluctuations of the frequency of the free wave and limitations of the linear growth of this wave due to nonlinear effects associated with the tendency of the nonequilibrium medium to go over to an equilibrium state. In the majority of cases, the first factor is the most dangerous, whereas the nonlinear limitation of the amplitude of the linearly unstable wave as a rule does not interfere with the further amplification of the weak signal wave; thus, the process will continue as long as these waves do not become of the same order of magnitude. Such a process was specifically investigated in detail as it applies to the interaction of waves in a beam with plasma [33].

3. Stabilization of Linear Instability due to Energy Transfer by Attenuating Waves

Such a process, which is of greatest interest for nonequilibrium media with a spectrally narrow instability domain (for example, parametrically excited media; a beam in a plasma) can be described by the system (II.1) for $\sigma_{1,2}\sigma_3 < 0$ and $\nu_i\nu_j < 0$. In the approximation based on random wave phases, one may write the equations for the intensities as follows (we assume that $\sigma_{1,2} = -\sigma_3$; $\nabla = 0$)

$$\dot{N}_{1,2} = \sigma^2 N_1 N_2 N_3\left(\frac{1}{N_1} + \frac{1}{N_2} - \frac{1}{N_3}\right) - \nu_{1,2} N_{1,2},$$
$$\dot{N}_3 = -\sigma^2 N_1 N_2 N_3\left(\frac{1}{N_1} + \frac{1}{N_2} - \frac{1}{N_3}\right) - \nu_3 N_3. \tag{II.5}$$

From (II.1) and (II.5) it follows directly in the case given that if one of the waves grows ($\nu_i < 0$), then the limitation of instability is possible only when the process is a decay process – stabilization of one-wave instability at the frequency $\omega_{1,2} < \omega_3$ is impossible, since for $t \to \infty$ the quasiparticles $\omega_{2,1}$ vanish, and the process of energy conversion upward along the spectrum ceases. It is just as obvious that for growth of both low-frequency waves having dynamic phases, the stabilization of the instability due to the attenuating wave ω_3 is possible only for equal actual growth rates of these waves – in a unit time an identical number of quanta ω_1 and ω_2 must be

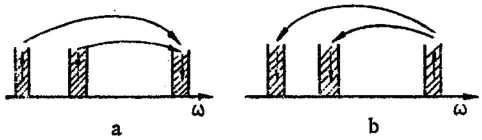


Fig. 19. Stabilization of the instability: a) for decay; b) for merging.

produced; otherwise, those quanta of which the greater number is produced per unit time will not have anything to merge with.

For interaction of waves having dynamic or random phases, the character of the constraint on the instability turns out to be different. When the phases are random, then it follows that if stabilization is possible an operating regime with a constant energy level in each of the waves is established:

$$N^0_{1,2} = \frac{|\nu_3|\,\nu_{2,1}}{\sigma^2(\nu_1+\nu_2-|\nu_3|)}, \qquad N^0_3 = \frac{\nu_1\nu_2}{\sigma^2(\nu_1+\nu_2-|\nu_3|)} \tag{II.6}$$

for the decay process (see Fig. 19a), and

$$N^0_{1,2} = \frac{\nu_3\,|\nu_{2,1}|}{\sigma^2(\nu_3-|\nu_1|-|\nu_2|)}, \qquad N^0_3 = \frac{|\nu_1\nu_2|}{\sigma^2(\nu_3-|\nu_1|-|\nu_2|)} \tag{II.7}$$

for the merging process (see Fig. 19b, $\nu_{1,2} < 0$, $\nu_3 > 0$).

Stabilization is possible when the resultant attenuation exceeds the resultant growth rate $\nu_1 + \nu_2 > |\nu_3|$, or when $\nu_3 > |\nu_1| + |\nu_2|$. These same conditions also guarantee, as can easily be checked, the stability of the stabilization regime for interaction of waves having random phases.

For interaction of waves having well-defined phases, the establishment of an operating regime with a constant energy distribution between waves as a result of stabilization of the instability is impossible – the stationary state in the system (II.1) for $\sigma_{1,2}\sigma_3 < 0$, $\nabla = 0$ and $\nu_{1,2} > 0$, $\nu_3 < 0$ or $\nu_{1,2} < 0$, $\nu_3 > 0$ exists but is always unstable. Only the trivial case of the generation of the second harmonic is an exception. However, it turns out to be possible to have a dynamic stabilization regime – the intensities of the interacting waves are limited but vary with time. Under these conditions, the energy flux from the instability domain to the drain domain turns out to be modulated. Using the example of decay interaction, let us consider this process in greater detail.*

4. Decay Triplet. Development of Stochasticity

Let us immediately emphasize the fact that whereas the classical problem of a conservative triplet ($\nu_{1,2,3} = 0$) serves as the elementary "block" in the theory of waves in equilibrium media, it follows that an investigation of a nonconservative triplet is one of the principal problems in the theory of wave interactions in nonequilibrium media.

Thus, let us consider the dynamics of a wave triplet in which high-frequency (hf) modes have a linear growth rate γ, while the low-frequency (lf) modes attenuate [34]:

$$\begin{aligned} \dot{A}_{1,2} &= -\sigma A_3 A_{2,1} \sin\Phi - \nu_{1,2} A_{1,2}, \\ \dot{A}_3 &= \sigma A_1 A_2 \sin\Phi + \tilde{\gamma} A_3, \\ \dot{\Phi} &= \sigma (A_1A_2A_3)^{-1}\left(A_1^2A_3^2 + A_2^2A_3^2 - A_1^2A_2^2\right)\cos\Phi - \Delta\omega, \end{aligned} \tag{II.8}$$

where $\Phi = \varphi_3 - \varphi_2 - \varphi_1$ is the phase difference of the wave.

a) Averaging over Elliptic Functions. First of all, let us consider the case of fairly large initial energies when the nonlinear terms in (II.8) substantially predominate over the linear terms.† Then the solution of our problem may be considered close to the solution of a conservative triplet,

*It may be shown that the stabilization of two growing waves having the dynamic phases due to attenuating high-frequency waves is possible only if the number of interacting waves exceeds three.
†These results were obtained in collaboration with E. N. Pelinovskii.

$$A_1 = s\,\Omega\,\mathrm{cn}\,(\Omega t, s),$$
$$A_2 = -\,\Omega\,\mathrm{dn}\,(\Omega t, s), \qquad \text{(II.9)}$$
$$A_3 = s\,\Omega\,\mathrm{sn}\,(\Omega t, s)$$

(sn z, cn z, dn z are elliptic Jacobi functions; s and Ω are arbitrary parameters), and by averaging over the elliptic functions one can determine the evolution of the parameters of the solution – the modulus s and the frequency Ω. It is simplest to do all this by making use of the energy relationships derived from (II.8):

$$\frac{d}{dt}\left(A_1^2 - A_2^2\right) = 2\left(\nu_2 A_2^2 - \nu_1 A_1^2\right); \qquad \text{(II.10a)}$$

$$\frac{d}{dt}\left(A_3^2 + A_1^2\right) = 2\left(\gamma A_3^2 - \nu_1 A_1^2\right). \qquad \text{(II.10b)}$$

Averaging these equations over the period of the elliptic functions, we obtain the following equations for s(t) and Ω(t):

$$s\frac{ds}{dt} = (\gamma + \nu_1) F(s, q); \qquad \text{(II.11)}$$

$$\frac{d\Omega}{dt} = (\gamma + \nu_2)\, G(p, s)\, \Omega, \qquad \text{(II.12)}$$

$$F(s, q) = 1 - s^2 - \frac{E(s)}{K(s)} + qs^2\frac{E(s)}{K(s)},$$

$$G(s, p) = p - \frac{E(s)}{K(s)}, \qquad q = \frac{\gamma + \nu_2}{\gamma + \nu_1}, \qquad p = \frac{\gamma}{\gamma + \nu_2}.$$

Here E and K are complete elliptic integrals with modulus s. These equations are valid if $A_1 < A_2$; otherwise, ν_1 and ν_2 are simply interchanged.

Equation (II.11) has three equilibrium states in the general case which correspond to vanishing of the function F(q, s): $s_1 = 0$, $s_2 = 1$, and $s_3 = s_0(q)$, where s_0 is the root of the equation $F(q, s_0) = 0$ and exists only for $1/2 \le q \le 1$. The values $q \ge 1$ correspond to the case $\nu_1 \le \nu_2$, while the values $q \le 1/2$ correspond to $\nu_1 \le \gamma + 2\nu_2$. For $q \le 1/2$ only the state $s = 0$, which in the given approximation corresponds to attenuation of the amplitudes of all the waves, is stable. In a practical situation, such attenuation will, of course, continue not to zero but only to the level at which the nonlinear terms in (II.8) become of the order of the linear terms and the averaging method is obviously inapplicable. For $q \ge 1$, the state $s = 1$ corresponding to the solution of (II.9) in the form of a single pulse (soliton) of the envelopes of the lf modes and a transfer pulse of the hf mode is stable. It is obvious that in the region of small fields the approach based on averaging is again inapplicable, and it turns out that in this important case it is necessary to investigate the solutions in some different manner [see items b) and c)]. For intermediate values $1/2 < q < 1$, the state $s = s_0(q)$ turns out to be stable; this corresponds to the establishment of a fully defined shape of the periodic variation of the envelope (this shape, of course, depends on the relationship between $\nu_{1,2}$ and γ). It follows from (II.12) that the amplitude of the beats under these conditions may not necessarily tend to stationary states but may grow slowly [under these conditions it follows that within the framework of the considered Eqs. (II.8), there is no stabilization], or it may attenuate (under these conditions, as has already been said, our "averaging" approach is not suitable). There are, however, specific relationships between $\nu_{1,2}$ and γ for which $G(p, s_0(q))$ in (II.12) vanishes; then there exists stationary periodic regimes for the variation of the envelope for which the amplitude of the beats depends on the initial conditions.

Thus, from the analysis presented it follows that notwithstanding the presence of small nonconservative terms in (II.8), the averaging method which appears natural here is suitable only in individual exotic situations or for the analysis of transients over finite time intervals.

b) Qualitative Analysis. Exact Synchronism. Let us now reject the assumption that the law governing the variation of the amplitude and phases of the interacting modes is close to (II.9), and let us use methods in the qualitative theory of differential equations. The analysis of the system (II.8) is substantially simplified for $\nu_1 = \nu_2 = \nu$, since under these conditions it follows from (II.10a) that

$$\left(A_1^2 - A_2^2\right) = \mathrm{const}\, e^{-\nu t} \to 0. \qquad \text{(II.13)}$$

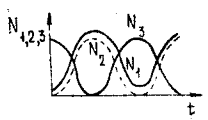


Fig. 20

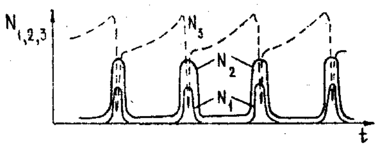


Fig. 21

Fig. 20. Exchange of energy between modes for three-wave interaction in an equilibrium medium.

Fig. 21. The proposed oscillograms of the intensities of attenuating (lf) and the amplified (hf) modes for resonance interaction.

and that the intensity of the lf modes, beginning with a certain time, may be considered equal: $A_1^2 = A_2^2 = A^2$. Then having made use of the substitution:

$$X = \frac{\sigma}{\nu} A_3 \sin \Phi, \quad Y = \frac{\sigma}{\nu} A_3 \cos \Phi, \quad Z = \frac{\sigma^2}{\nu^2} A^2,$$
$$\tau = \nu t, \quad \delta = \Delta\omega/\nu, \quad \gamma = \tilde{\gamma}/\nu, \tag{II.14}$$

we shall have the following result instead of (II.8):

$$\begin{aligned} \dot{X} &= Z + \delta Y - 2Y^2 + \gamma X, \\ \dot{Y} &= -\delta X + 2XY + \delta Y, \\ \dot{Z} &= -2Z(X + 1). \end{aligned} \tag{II.15}$$

For an analysis of this system we make use of phase space. It is evident that its structure in the given case is determined by two parameters – the growth rate γ of the hf wave and the detuning from exact synchronism δ. The stabilization of the hf mode by attenuating low-frequency modes, obviously, is possible only for a fairly small γ.* Assuming further that $\gamma \ll 1$, we shall consider it as a small parameter. Under these conditions, the phase space splits into domains of fast and slow motions. The latter are situated near the straight line $Y = \delta/2$, $Z = 0$, and are determined by the inequalities

$$|Z| \lesssim \gamma |X|, \qquad |2Y^2 - \delta Y| \lesssim \gamma |X|. \tag{II.16}$$

The system (II.11) has only two equilibrium states, and both of them are unstable (of the "saddle-focus" type). One is situated at the origin (for $\delta = 0$ it goes over into a "saddle-node"), while the other, which has the coordinates $X = -1$, $Y = \delta/(2 - \gamma)$, $Z = \gamma[1 + \delta^2/(2 - \gamma)^2]$, is situated on the boundary between fast and slow motions.

In the conservative case, three-mode interaction is a periodic alternation of the merging and decay processes (see Fig. 20). What changes when losses exist for merging lf modes? It is obvious that the time between the end of the merging process and the beginning of the decay process (the latter consists of dissipation of weak lf modes) slows down the manifestation of the decay growth rate at low amplitudes of the lf modes. In the presence of a small growth rate, the hf mode will increase exponentially during this time. Thus, for small γ the oscillograms of the amplitudes of the lf modes will be converted into a sequence of widely spaced peaks (solitons), while the hf modes will be converted into a saw with experimental peaks (see Fig. 21).

<u>Exact Synchronism ($\delta = 0$).</u> In the phase space of the system (II.11) there are two integral surfaces (their trajectories do not intersect) $Z = 0$ and $Y = 0$ for $\delta = 0$; cones of slow motions are seated on the intersection of these planes (see Fig. 22). The phase diagrams of these surfaces are displayed in Fig. 23a, b. On the $Z = 0$ plane, the domain of slow motions is bounded by the parabola $\gamma|X| = 2Y^2$, while for $X > 0$ it is the isocline of horizontal tangents. Outside this domain, the effect of a small growth rate is insubstantial, and the trajectories are practically indistinguishable from the circles $X^2 + Y^2 = \text{const}$ on which energy is conserved. The motions on this plane are asymptotically stable relative to perturbations of Z in the domain $X > -1$.

Motions on the integral plane $Y = 0$ are asymptotically stable relative to perturbations of Y in the $X < -\gamma/2$ domain. On this plane, the domain of slow motions is bounded by the straight lines $Z = \pm\gamma X$; for $X < 0$

*If one makes use of the analogy with quasiparticles, then $\gamma < 2$ must hold; in a numerical experiment, the limitations of the instability was observed for $\gamma < 1/6$ (see below).

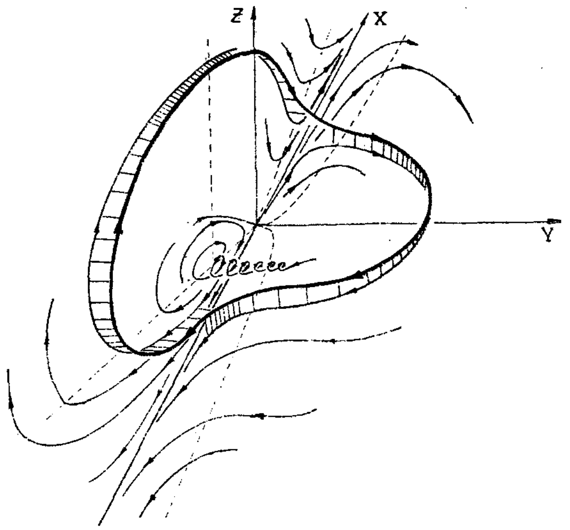


Fig. 22. The phase space of the system (II.11).

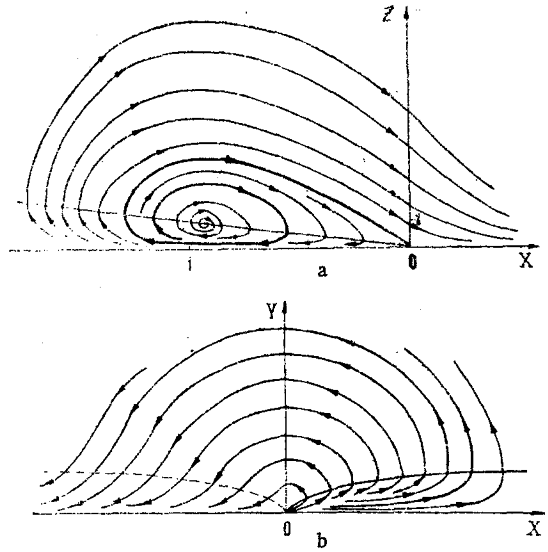


Fig. 23. Phase diagrams of the integral planes for $\delta = 0$: a) $Z = 0$; b) $Y = 0$.

this is the isocline of vertical tangency (see Fig. 23b). On these integral surfaces $Y = 0$ and $Z = 0$, the trajectories behave in a very similar manner – for $t \to \infty$ they squeeze asymptotically toward the X axis.

Let us now represent the entirety of the character of the motions which correspond to trajectories which are not too far from the $Z = 0$ and $Y = 0$ planes. While squeezing toward the $Z = 0$ plane in the domain $X > -1$, the image point moves along a circle until it enters the domain of slow motions (see Fig. 23a) where the amplitude of the hf mode grows exponentially. After emergence from the domain of slow motions, it rises while remaining close to the $Y = 0$ plane and departs in the direction of increasing X (the decay state), after which it crosses the $X = 0$ plane and descends to the $Z = 0$ plane (merging). Having landed in the domain of slow motions here, the point again moves in the neighborhood of $Z = 0$ (X grows) and then goes over onto the circle $X^2 + Y^2 = \text{const}$, etc.*

In order to determine the character of the considered motions, let us clarify how the points of the narrow vertical strip $0 < Y_1 < Y < Y_2 \ll 1$ are mapped on the $X = 0$ plane by the phase trajectories for $X > 0$ into points on the horizontal strip $0 < Z_1 < Z < Z_2 \ll 1$ of this same plane, and then how the trajectories for $X < 0$ again map them into points of the vertical strip. This yields the dependence of the parameters of motion at the end of a period on their values and beginning of the period. Then using an iteration procedure, it will be possible to establish such qualitative features of the process as the existence of periodic operating regimes and the development of stochasticity.

The qualitative form of the mapping of the vertical strip into the horizontal strip and then back into the vertical strip is displayed in Fig. 24. Here the successive transformations of a certain line in the vertical

*For large initial energies, processes are also possible which are such that the point circumvents the domains of slow motions and enters the domain $Z > \gamma|X|$ for $X < 0$ and small Y immediately by detaching itself from the circle for $X < -1$. However, for small values of γ these motions must be unstable – the energy losses of the hf modes during the merging and decay stages are not compensated by the amplification of the hf mode.

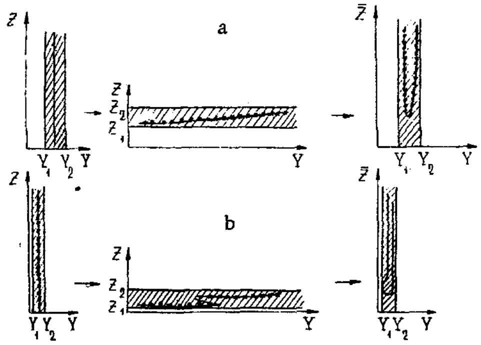


Fig. 24. Qualitative form of the mapping of the points of the vertical strip of the $X = 0$ plane into a horizontal strip and the horizontal strip back into a vertical strip.

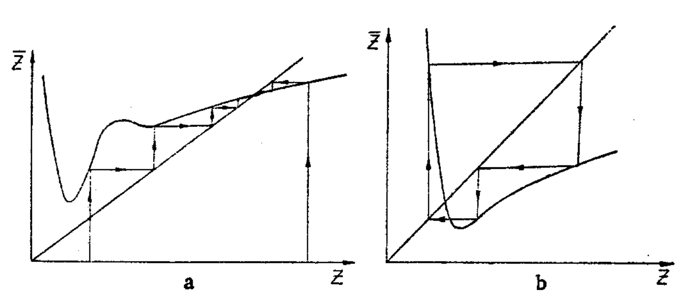


Fig. 25. Point mappings of the straight line Z into itself: a) the initial conditions on the X = 0 plane lie in the strip $0 < Y < Y_1$; b) they lie in the strip $Y_1 < Y < Y_2 \ll 1$.

strip are depicted for two different cases, depending on the closeness of the vertical strip to the Z axis.

Let us clarify the diagram of the mapping for the more complicated case depicted in Fig. 24b. From (II.11) it follows that the higher the trajectory intersects the vertical strip in terms of the value of Z, the further from the X axis it descends onto the Z = 0 plane. The trajectory which begins on the X = 0 plane for fairly large Z does not land in the domain of slow motions after it has descended onto the Z = 0 plane, but moves immediately along the circle $X^2 + Y^2 = R^2$. It is clear that another trajectory also arrives in the vicinity of this same circle; this trajectory, after having passed through the vertical strip at small Z, descends onto the Z = 0 plane in the domain of slow motions and departs from that domain [intersecting the parabola – see Fig. 23a] at $X^2 \approx R^2$. Thus, the trajectories which intersect the vertical strip with a great distance between them turn out to be close together when they intersect the horizontal strip, and vice versa. It is these facts which explain the development of the "horseshoe" in the mapping patterns.

An analysis similar to the one carried out above is also applicable to trajectories in the domain X < 0 which departs from the Z = 0 plane and squeeze toward the Y = 0 plane; however, here the role of the parabola is played by the straight line $Z = -\gamma X$ (i.e., the character of the mapping of the horizontal strip into the vertical strip is similar to the previous one). From this it follows that a double horseshoe may developed in the overall mapping diagram. For multiple passage through the X = 0 plane, which corresponds to multiple application of mapping, the pattern of the motion may become very complex and confused.

Assuming that the considered motions are stable, let us investigate their structure in greater detail. For this purpose, let us coarsen the model by identifying points having identical Z in the vertical strip and points having identical Y in the horizontal strip. Under these conditions, we obtain a mapping of a straight line Z into itself instead of the mapping of a strip into a strip [i.e., we obtain the dependence of subsequent points $\bar{Z}$ on initial points $Z : \bar{Z}(Z)$ (see Fig. 25)]. For different positions of the $\bar{Z}(Z)$ curve relative to the bisectrix $\bar{Z} = Z$, either a unique stable one-period regime (see Fig. 25a) or a set of regimes with modulation (multiperiod regimes) corresponding to both cycles on the $\bar{Z}$, Z plane exists (see Fig. 25b).*

Further, having made use of the well-known results of the formal theory of point-to-point mappings [35, 36], one may state that if there exists periodic motion with an odd number of periods (such as, for example, the three-period motion depicted in Fig. 25b), then regardless of its stability there exists an additional infinite (uncountable) set of unstable multiperiod motions and a finite (countable) number of stable ones exist in the system. Thus, the structure of the considered motions may actually turn out to be extremely complicated.

c) Effect of Detuning on Exact Synchronism.† In a conservative triplet the detuning, as is well known, decreases the degree of energy exchanged between nodes (for the same initial conditions). In our (nonconservative) case the same thing occurs – i.e., the maximum attainable values of Z are diminished for X = –1.

*The curve $\bar{Z}(Z)$ is deformed and is lowered or raised as a function of the closeness of the vertical strip compressed into a line to the Y = 0 plane (or of the horizontal strip to the Z = 0 plane); i.e., the form of this "coarsened" mapping depends on the initial conditions on the X = 0 plane. This is the "payment" for reducing the dimensionality of the mapping.

† For introduction of detuning the phase space of the system (II.15) becomes crude – the integral surface Y = 0 disintegrates, while the equilibrium state at the origin is converted from a saddle-node into a saddle-focus (see Fig. 26).

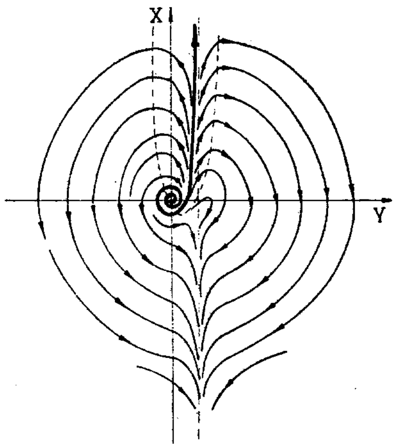


Fig. 26. Phase diagram of the integral plane Z = 0 for $\delta > 0$.

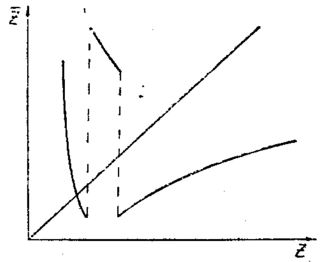


Fig. 27. Discontinuous mapping of the straight line Z into itself.

The latter fact means that situations are possible in which the line which is transferred during the mapping process by the trajectories from the X = 0 plane to the neighborhood of the Y = 0 plane "settles" on this plane near the equilibrium state X = −1, Z = γ for X < 0. This means that the considered mapping is split into two classes of motions by a separatrix which enters the zero equilibrium state. The trajectories which enter the neighborhood of the Y = 0 plane intersect the X = 0 plane outside the unwinding separatrix (see Fig. 22) and move further in the direction of the growth of X. The same trajectories which fall inside the separatrix turn toward decreasing X before they intersect the X = 0 plane. They intersect this plane, but only after one or several revolutions in the X < 0 domain and at values of Z which are already large. The diagram of the "coarsened" mapping of a straight line into a straight line corresponding to the situation described is displayed in Fig. 27. The discontinuities on the $\bar{Z}(Z)$ curves in the region of small $\bar{Z}$ are precisely those which correspond to the separation of the motions by separatrix. For dynamic systems which can be described by such a piecewise-smooth transformation, one may prove a very strong statement concerning the existence of mixing in this system for certain constraints [37].

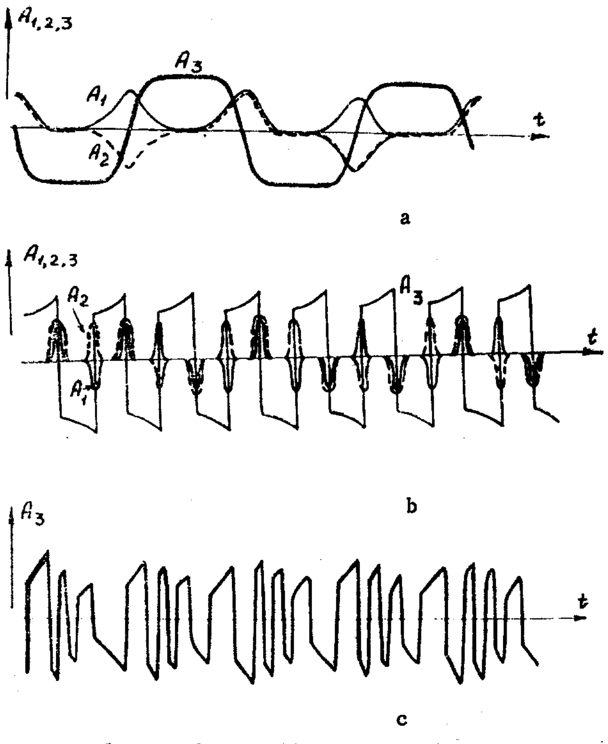


Fig. 28. Exchange of energy between modes in the resonance case: a) one-period regime for $\gamma = 0.15$; b) one-period regime for $\gamma \doteq 0.1$; c) three-period regime for $\gamma = 0.1$.

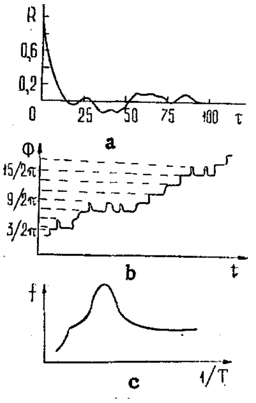


Fig. 29

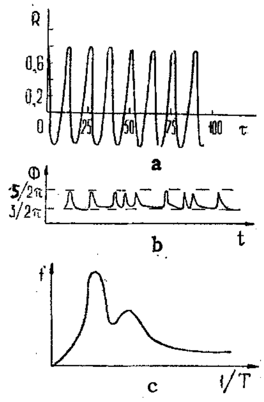


Fig. 30

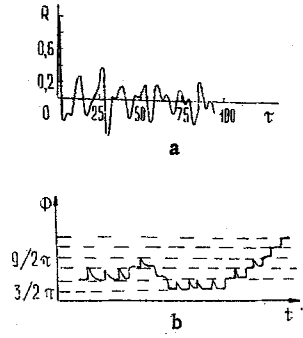


Fig. 31

Fig. 29. Characteristics of the total-stochasticity regime: a) autocorrelation function of the realization $A_3(t)$; b) the dependence $\Phi(t)$ – accumulation of Φ can be seen; c) distribution function of the interval between jumps according to their reciprocal duration $f(1/T)$.

Fig. 30. Characteristics of the partial-stochasticity regime: a) autocorrelation function; b) oscillogram $\Phi(t)$ – there is no accumulation of Φ; c) distribution function.

Fig. 31. Transitional stochasticity: a) autocorrelation function; b) oscillogram $\Phi(t)$.

Thus, the analysis which has been performed demonstrates that if the regime in which an unstable mode is stabilized due to decay into attenuating modes is possible, then in the presence of detuning it may be stochastic (this is in a purely dynamic system!).

5. Decay Tripet. Computer Experiments

The detailed numerical experiments performed in [34, 38] with the system (II.8) has confirmed all of the principal results of the qualitative analysis. Let us say immediately that the dynamic processes were observed only in the absence of detuning $\delta = 0$, while stochastic processes, on the contrary, were observed only for $\delta \neq 0$. On the order of 100 realizations were obtained (of them 2/3 were for $\delta \neq 0$) for triplet values of the parameters $\gamma = 0.15$; 0.1; 0.01, $\delta = 0$; 0.001; 0.01; 0.1 and of the initial conditions $A_{1,2,3} = 5$–10, $\Phi(0) = 0$–2π.

a) In the resonance case $\delta = 0$ two forms of motion were observed, depending on the initial energies: one-period motions (see Fig. 28a, b) for which the maximum values of A_3 were 13-18, and multiperiod motions (see Fig. 28c). Under these conditions only three- and four-period motions were observed to be stable, while others (for example, five- and six-period motions) had a small stability domain and went over into each other from time to time. The amplitude of the multiperiod motions was noticeably lower: $A_{3\,max} \sim 5$–8, while the period was 1.5-3 times longer than that of the one-period motions. Stabilization of the unstable mode $\omega_3 = \omega_1 + \omega_2$ was possible only for a fairly large relative attenuation and a low growth rate: namely, for $\gamma < 1/6$. For $\gamma \sim 1/6$, the shape of the oscillations was described approximately by elliptic functions (see Fig. 28a), while for a decrease in γ it became more and more relaxational – even for $\gamma = 0.1$ the motions were in the form of discontinuous exponentials for A_3 and narrow solitons for A_1 and A_2 (see Fig. 28b; i.e., clearly defined fast and slow motions were observed). The period of the oscillations was ~1-1.5, while the time required for fast motions was < 0.1 [with the exception of such time intervals, the phase Φ remained equal to approximately $\pm\pi/2$ regardless of the value of $\Phi(0)$].

b) In the nonresonance case, for which the conditions for synchronism between modes were inexactly satisfied ($\delta \neq 0$), all of the observed motions were stochastic. Their properties, unlike those of dynamic processes, depended substantially on the initial conditions for Φ. For $\delta > 0$, $A_{1,2,3} \sim 5$–10, and $\Phi = 0$–2π, three qualitatively different groups of stochastic regimes were observed – their autocorrelation functions are displayed in Figs. 29a, 30a, and 31a. The same figures give the distribution functions of the interval between jumps according to their reciprocal durations $f(1/T)$ and the oscillograms of the phase differences $\Phi = \Phi(t)$, which correspond to these regimes. The amplitude realizations corresponding to the different stochastic regimes were qualitatively similar, and only the maximum values of the amplitudes vary. One of these typical

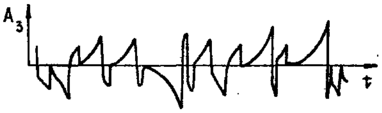


Fig. 32. Oscillogram of the amplitude of the unstable model in the stabilization regime for $\delta = 0.01$.

realizations – the dependence of the amplitudes of an unstable mode on time – is displayed in Fig. 32. The regions of initial values of $\Phi(0)$ for which these different regimes were observed (we shall arbitrarily call them "total stochasticity" – Fig. 29, "partial stochasticity" – Fig. 30, and "transitional stochasticity" – Fig. 31) are depicted on the X, Y plane (see Fig. 33). Let us add the fact that the average period (the time required for slow motions of all of the stochastic processes) was one order of magnitude greater than the time required for slow processes in the dynamic case.

Let us now discuss the results of the experiment from the point of view of the theory expounded above. Let us recall that the stochastic regimes were observed only when a detuning δ was introduced, it being true that even for a small δ the intensities of these two motions Z_{max} or $(X^2 + Y^2)_{max}$ were approximately one order of magnitude smaller than for $\delta = 0$. Thus, the statement used in the qualitative analysis to the effect that the detuning reduces the level of the maximally attainable Z was confirmed [it was on the basis of this statement that the discontinuous mapping $\overline{Z}(Z)$ was obtained].

As it followed from a qualitative analysis, disordered motions of two kinds are possible in the system (3) for $\delta > 0$. Trajectories in the right half-space correspond to one of them – for them Y is always greater than $\delta/2$, and the accumulation of the phase difference Φ is impossible for periodic or quasiperiodic motion. The other form of motion corresponds to trajectories in the left half-space $Y < \delta/2$; for these motions the accumulation of the phase difference is possible, since the trajectories may enter from the domain $Y < 0$ into the layer $0 < Y < \delta/2$ and, consequently, encompass the Z axis.

It is not difficult to confirm the fact that the experimentally observed "partial-stochasticity" regime is precisely the one that corresponds to "right" disordered motions and those "left" ones whose trajectories do not enter the layer $0 < Y < \delta/2$. Actually, during the slow stage of "right" motion the phase difference Φ is close to $\pm\pi/2$ or to $\pm(3/2)\pi$; then it varies rapidly by the amount π – motion near the $Y = \delta/2 + 0$ plane, after which it again adopts its previous value – motion near the $Z = 0$ plane outside the domain of slow motions. An analogous situation also obtained for "left" motions lying in the half-space $Y < 0$. It is precisely such a picture of the time variation that was also observed experimentally in the "partial-stochasticity" regime (see Fig. 30b).

However, during the process of "left" motion, for which the image point enters the layer $0 < Y < \delta/2$ from the half-space $Y < 0$, the accumulation is already impossible – Φ varies by the amount π during the process of fast motion in this layer, and then the trajectory again exits into the left half-space and moves near the circle, while Φ increases by another π. It is this which explains the accumulation of Φ that is observed in the "total-stochasticity" regime (see Fig. 29b).

Let us emphasize the fact that the properties of the "total" and "partial" stochasticity regimes (for example, the form of the autocorrelation function) did not depend on the calculation accuracy over wide limits. This proves the dynamic origin of such stochasticity in the computer experiments. Only the "thickness" of the boundary separating the existence domains of these regimes (see Fig. 33) depended on the calculation accuracy

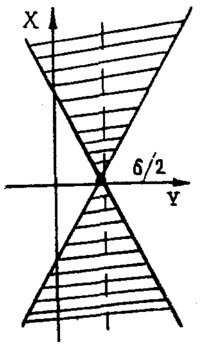


Fig. 33. Regions of initial conditions on the $Z = 0$ plane which correspond to the different stochastic regimes.

(actually, on the fluctuation level) within small limits. On the actual boundary, the stochasticity regime displayed in Fig. 31 was observed. Its properties depended on the calculation accuracy, and it evidently does not characterize the intrinsic stochasticity of the dynamic system.

The experiments that were carried out in [34] on the degenerate interaction of the stable ω and unstable $(2\omega + \Delta\omega)$ modes [such interaction is likewise described by the system (II.11)] for inexact synchronism ($\omega \gg \Delta\omega > 0$) coincide completely with the results expounded and thereby confirm the correctness of the model chosen [i.e., the validity of the replacement of the system (II.8) by the system (II.11) in the qualitative analysis].

Concluding the present section, we again focus attention on the fact that the development of the stochastic behavior in a nonlinear nonequilibrium medium is not necessarily associated with a large number of interactions. The disordered behavior which requires statistical methods for its description† may already develop in a system of three and even two coupled waves (modes).

6. Resonance Interaction of Nonincreasing Modes with a "Self-oscillatory" Mode

In connection with the analysis of three-wave processes in a nonequilibrium medium having an "equilibrium" ($\sigma_{1,2,3} = \sigma^*_{1,2,3}$, $\sigma_{1,2}\sigma_3 < 0$) nonlinearity it is worth also mentioning the interaction of the amplified and nonincreasing waves in those cases when stabilization of the instability is accomplished as a result of the nonlinear viscosity that enters the game at large amplitudes of the linearly amplified waves.‡ Such viscosity may specifically be a consequence of the nonlinear distortion of the growing sinusoidal wave as a result of its generation of attenuating harmonics. In the "Raman-scattering" approximation, where the amplitudes of the higher harmonics emulate the amplitude of the first harmonic, the equation $-\dot{a} + va' = \gamma a - \alpha a|a|^2$ is obtained for the latter. The interaction of such a "self-oscillatory" mode with two others in a medium having a quadratic nonlinearity can be described by the system

$$\begin{aligned} &\dot{a}_1 + v_1 a_1' = -a_3 a_2^* + \gamma a_1 - \alpha a_1 |a_1|^2, \\ &\dot{a}_2 + v_2 a_2' = -a_3 a_1^*; \quad \dot{a}_3 + v_3 a_3' = a_1 a_2. \end{aligned} \tag{II.17}$$

As was demonstrated in [40], the energy flux from the unstable mode a_1 into a_2 and a_3 turns out to be modulated (in time for spatially uniform fields and in space for $e^{i\omega t}$ fields when interaction takes place along x). Under these conditions, the amplitudes of the modes oscillate, while the phase difference performs "jumps." Here the situation is similar to that observed in a resonance triplet for unstable and attenuating modes in the case of complete synchronism (see Paras. 4 and 5).

Note that such a process is realized in parametric oscillators with internal pumping [41] where the pumping wave is generated directly in the resonator in which parametric conversion takes place.

III. EXPLOSIVE INTERACTION OF WAVES

We have already said that in a nonequilibrium medium resonance interaction of waves may lead to an explosive instability which is more rapid than linear instability; under these conditions, the amplitudes of all of the waves increase simultaneously and, under the idealizations usually employed, go to infinity in a finite time (see Para. 1 of Sec. II). Formally, the "explosion" is guaranteed by the aggregate nature of the coefficients of nonlinear interaction σ, provided that $\max\{\theta_i\} - \min\{\theta_i\} < \pi$ [see (II.2)]. However, the physical explosion mechanisms may be fairly varied and depend on the nature of the wave and the character of the nonequilibrium of the medium.

1. Interaction of Waves Having Energies of Different Signs. A Beam in a Plasma

First of all, let us focus attention on the apparent obvious asymmetry of the resonance conditions for the frequencies and wave numbers – the synchronism conditions. It is well known that in equilibrium media the

† For conservation systems, the development of stochasticity for a number of degrees of freedom n = 1.5 is well known [39].

‡ Such stabilization of the unstable mode is observed, for example, for Tollmien – Schlichting waves in certain hydrodynamic flows. The increasing wave changes the profile of the main flow, as a consequence of which the growth rate is reduced.

transfer of energy is possible only from the highest frequency wave to the low-frequency waves but not the other way; at the same time, there are no such constraints on the quasimomenta of the wave. For example, a wave with a very small and even a zero (a spatially uniform field) momentum may also decay and form a wave with arbitrarily large but oppositely directed momenta. It is clear that this asymmetry is associated with the physical reality of negative wave numbers (they correspond to oppositely traveling waves) and the apparent unreality of negative frequencies. If in some medium waves having negative frequencies were to have a physical meaning and were actually to exist, then in that medium the constraints on the character of the nonlinear interaction which are valid for equilibrium media would turn out to be violated.

Since the frequency corresponds to the energy of the quasiparticles, one may henceforth consider all of the frequencies to be positive, and the waves having a negative frequency may be assigned a negative energy sign. In their physical meaning, waves having a negative energy are waves which are such that the resultant energy of the "medium – wave" system decreases with increasing amplitude. Such waves may exist in inhomogeneous media [42] or in a medium made up of inversely populated particles [43]. Longitudinal electrostatic waves whose spectrum is situated in the region of anomalous dispersion of the medium $\partial\varepsilon/\partial\omega < 0$ also have negative energy; for them the energy is $W = (\omega/8\pi)(\partial\varepsilon/\partial\omega)\langle E\rangle < 0$.

Let us clarify the meaning of the concept of "negative energy" using the example of space-charge waves in a moving electron beam which can be described by the equations

$$\begin{gathered}\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial x}-\frac{e}{m}E=0,\\ \frac{\partial n}{\partial t}+N\frac{\partial u}{\partial x}+V_0\frac{\partial n}{\partial x}=0,\\ \frac{\partial E}{\partial x}-4\pi e(n-N)=0,\\ \omega_0^2=\frac{4\pi Ne^2}{m}.\end{gathered} \tag{III.1}$$

The dispersion equation of such a medium is

$$\varepsilon(\omega, k)=1-\frac{\omega_0^2}{(\omega-kV_0)^2}. \tag{III.2}$$

It is evident that on the branch $\omega = kV_0 - \omega_0$, which corresponds to the slow wave, $\partial\varepsilon/\partial\omega = 2\omega_0^2/(\omega - kV_0)^3 = -2/\omega_0 < 0$ (i.e., the energy of the slow wave is negative). This is associated with the following. In accordance with (III.1), the velocity perturbations are in phase with the density perturbations in a fast wave ($\omega = kV_0 + \omega_0$), while in a slow wave they are in phase opposition [44]. Consequently, for a fast wave the condensation segments have a velocity exceeding V_0, while the rarefaction segments have a velocity lower than V_0. Therefore, when a fast wave is excited in the beam, the resultant kinetic energy transferred by the beam exceeds the energy transferred by the unperturbed beam. However, if a slow wave is excited, then the opposite is true: The bunches travel more slowly than V_0, while the rarefactions travel faster. As a result, the kinetic energy transferred by such a beam is lower than the energy of a beam containing no waves.

For transverse electromagnetic waves, the energy may be negative (for example, in a medium consisting of two-level particles). Actually, in this case [43] we have

$$\varepsilon=1-\frac{\omega_0^2N_{12}}{\omega^2-\omega_{12}^2+2i\gamma\omega_{12}}, \tag{III.3}$$

where ω_{12} is the transition frequency; $\omega_0^2 = (4\pi e^2N/m)d$ (d characterizes the coupling of the particles with the field); $N_{12} = (n_1 - n_2)/n_2$; $n_{1,2}$ are the populations of the lower and upper levels. The energy of the wave at the frequency ω, where $\omega - \omega_{12} \gg \gamma$, is approximately equal to

$$\frac{\partial}{\partial\omega}(\omega^2\varepsilon)=2\omega\left[1+\frac{\omega_{12}^2N_{12}\omega_0^2}{(\omega^2-\omega_0^2)^2}\right] \tag{III.4}$$

and may be negative if the medium is inverted (the upper level is populated more heavily than the lower level). In accordance with (III.4), the energy of waves having the frequency

$$\omega-\omega_{12}<\frac{\omega_0}{2}\sqrt{n_2-n_1} \tag{III.5}$$

will be negative.

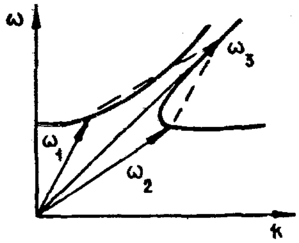


Fig. 34. The synchronism condition illustrating the possibility of an explosion in a beam – plasma system.

It is clear that the interaction of waves having a negative energy with an absorptive medium or with waves having a positive energy must be accompanied by instability – in giving up energy, such a wave will increase in amplitude.

The equations describing the interaction of waves having energies of different signs in a medium having a conservative nonlinearity that is quadratic in the field are written in the form ($\omega_3 = \omega_1 + \omega_2$)

$$\begin{aligned} s_{1,2}\left(\frac{\partial a_{1,2}}{\partial t} + v_{1,2}\frac{\partial a_{1,2}}{\partial x}\right) &= -\sigma_{1,2}\, a_3\, a_{2,1}^{*} \\ s_3\left(\frac{\partial a_3}{\partial t} + v_3\frac{\partial a_3}{\partial x}\right) &= \sigma_3\, a_1\, a_2 \end{aligned} \qquad (\sigma_i = \sigma_i^{*}), \qquad \text{(III.6)}$$

where s_i are the signs of the wave energies. If one of the low-frequency waves (for example, ω_1) has a negative energy, then the waves having the frequencies ω_3 and ω_2 interchange their roles, and now only the low-frequency wave ω_2 may decay. Explosion instability corresponds to the case when the high-frequency (ω_3) wave has a negative energy – under these conditions (III.6) can be reduced to (II.2), where $\theta_{1,2,3} = 0$. Namely, such a situation is realized in a plasma – beam system for interaction of a slow beam wave with two plasma waves [4] (see Fig. 34). Explosion instability in a plasma with a beam was observed experimentally in [45].

2. Interaction of Tollmien – Schlichting Waves in a Boundary Layer

Explosion instability may also develop for resonance interaction of waves having a positive energy; for example, in a medium whose nonlinearity is determined by the imaginary part of the permittivity (see below) or in hydrodynamic flows with a nonuniform velocity profile, an example of which may be found in a boundary layer.

For Reynolds numbers $R = U\delta/\nu > R_{cr}$, weakly unstable Tollmien – Schlichting waves (TS waves) are excited in the boundary layer [46]; the growth rate and structure of these waves across the flow is determined from the solution of the linear boundary-value problem† [47].

For $R > R_{cr}$, these are surface waves which travel along the flow (see Fig. 35). With a growth of R, increasing or weakly attenuating waves appear which also travel at an angle with respect to the flow. They were observed experimentally [48] for a fairly large "pumping" (the "pumping" is represented by the amplitude of the two-dimensional TS wave; i.e., two-dimensional waves turn out to be unstable relative to three-dimensional perturbations).

From an analysis of the dispersion equation for three-dimensional TS waves, it follows [49] that the two-dimensional wave having the frequency ω may be a synchronism with two three-dimensional waves having the frequency $\omega/2$ propagating at equal but opposite angles relative to the main flow (see Fig. 36). All these three waves have an identical phase velocity $v = \omega/k_x$ along the flow and, consequently, a common critical layer near y_0 – a layer where the phase velocity of the waves coincide with the velocity of the flow $U(y_0)$. It is precisely in the neighborhood of the critical layer that the nonlinear interaction of the waves forming the triplet is strongest.

The process of resonance interaction of TS waves can likewise be described by the system (II.1), where σ_i are complex and depend on the unperturbed velocity profile in the boundary layer. From symmetry concepts, it is obvious that $\sigma_1 = \sigma_2$. The specific form of σ_i for the model velocity profile

† Usually, it is called the Orr – Sommerfeld problem.

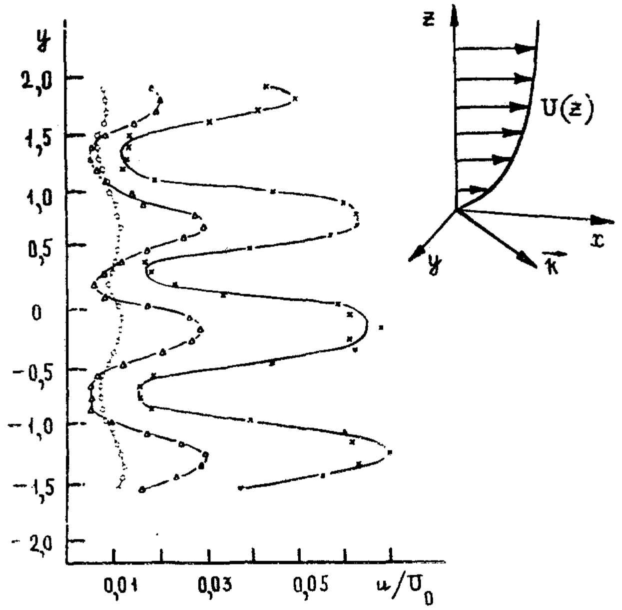


Fig. 35. Tollmien–Schlichting waves in a boundary layer.

$$U(y) = \begin{cases} y & (0 < y < 1) \\ 1 & (y > 1) \end{cases}$$

was found in [49], while for an actual boundary layer on a flat plate (a Blasius profile) it was found in [50]. In view of $\sigma_1 = \sigma_2$, an equation that is somewhat simpler than (II.2) is found here for the complex amplitude:

$$\dot{a}_{1,2} = a_3 a^*_{2,1}, \quad \dot{a}_3 = e^{i\theta} a_1 a_2. \tag{III.7}$$

For a change in the velocity profile in the boundary layer, the quantity θ changes and, consequently, the character of the interaction of TS waves likewise changes. For $\theta = \pi$, we have the conventional "exchange" interaction, while for $\theta \neq \pi$ an explosive growth of the amplitudes is possible:

$$\begin{aligned} |a_2| = |a_1| &= \left[\frac{\cos\Phi_0}{\cos(\theta - \Phi_0)}\right]^{1/2} \frac{A_0}{1 - tA_0\cos\Phi_0}, \\ |a_3| &= \frac{A_0}{1 - tA_0\cos\Phi_0}, \end{aligned} \tag{III.8}$$

where the steady-state phase difference between the waves $\Phi_0 = \arg a_3 - \arg a_2 - \arg a_1$ is a root of the equation $\tan\Phi_0 = \frac{1}{2}\tan(\theta - \Phi_0)$. Such a process of simultaneous growth of the harmonic was observed in experiments with an unstable boundary layer on a flat plate – see Fig. 37 [48].

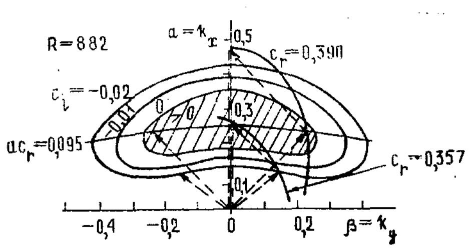


Fig. 36

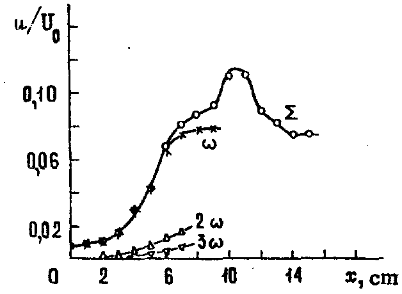
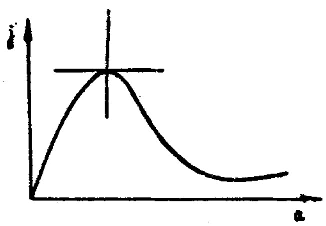


Fig. 37

Fig. 36. Synchronism conditions for a two-dimensional and two three-dimensional TS waves.

Fig. 37. Simultaneous growth of the harmonics of TS waves in a boundary layer on a flat plate.



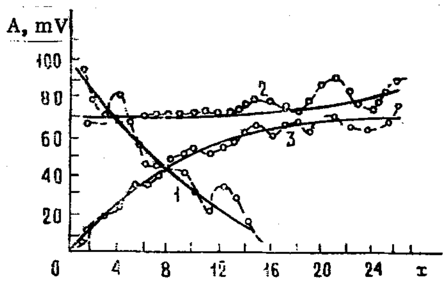


Fig. 38 Fig. 39

Fig. 38. Dependence of the conductivity on the field for tunnel transitions and Gunn semiconductors.

Fig. 39. Explosion instability in space for the interaction of the wave and its second harmonic in an LC line with tunnel diodes.

This process, however, cannot go far enough, since for a boundary layer having no "drainage" the Reynolds number increases downward along the flow, and secondary instabilities develop – the waves considered excite perturbations having higher frequencies. As a result, developed turbulence of the boundary layer is established.

Let us add that in accordance with a linear analysis the conditions governing resonance coupling may be fulfilled immediately for several wave triplets (see Fig. 36); i.e., turbulence may develop even before the development of secondary hf instabilities (see Para. 3, Sec. IV).

3. Explosion Instability in an Active Medium

The simultaneous growth of the amplitude of resonance-coupled waves having a positive energy is also possible in active media – specifically, in media such that the nonlinear conductivity has a decreasing segment (see Fig. 38) – distributed tunnel transitions [5, 51], Gunn semiconductors [52], etc. Such a medium may be considered quadratic (the imaginary part of the permittivity is proportional to the field) if the operating point is placed at the peak of the characteristic. It is easy to confirm the development of an explosion as follows. In an equilibrium quadratic medium, where the real part of ε ($\varepsilon = \varepsilon_H u$) is proportional to the field, the equation for the amplitudes of the coupled waves may be written in such a way that all of the coefficients of nonlinear interaction will be purely imaginary and of the same sign $a_{1,2,3} \sim i\varepsilon_H$. Assuming now that the nonlinearity is imaginary, we obviously arrive at Eqs. (II.2); for $\theta_{1,2,3} = 0$, which describes an explosion, the coefficients are real and of the same sign.

The physical cause of the "explosion" is conveniently clarified for the case of degenerate interaction: $\omega_1 + \omega_2 = \omega_3$ [$a_1 \equiv a_2$ in (II.2)]. For a harmonic wave, the amplification in a medium having the current density $j(u) = -\sigma u^2$ is absent on the average over a period – during one half-cycle the medium has positive conductivity, while during the other it has negative conductivity. For oscillations which are nonsymmetrical over a period (such as the sum of two harmonics) this is not so – the nonlinear amplification may exceed the absorption, and the amplitude of the two harmonics will increase simultaneously.

Such a growth was observed experimentally [51] in a two-wire transmission line with nonlinear leakage; the dependence of the current on the voltage in such a line has the form shown in Fig. 38. Tunnel diodes whose operating point was brought out onto the peak of the characteristic were used as the leakage elements. In Fig. 39 it is evident that the amplitudes of the interacting waves increase simultaneously along the line (curves 2, 3). However, if the conditions for synchronism between them were disrupted, then the amplitudes of the waves are not changed, or, when attenuation was introduced into the line they decreased along x (see curve 1). An analogous process was also observed for nondegenerate interaction when a third wave at the combination frequency develops in the presence of two waves; the generation of this combination-frequency wave was accompanied by a growth (!) of all three waves.

4. Interaction of Pulses and Beams during Explosive Instability

If the width of the beam or the duration of the pulses is less than or of the order of the characteristic duration or length of their interaction, then they can no longer be treated as spatially homogeneous, and it is necessary to take account of the finiteness of their propagation velocity. The group dissynchronism which develops in the general case may be reduced, for example, to a situation in which before the pulses or wave beams interact they have time to disperse in space. However, even for a great difference in velocities, this may not occur.

The analytic difficulties which develop in solving such problems can be overcome only in individual cases – specifically in analyzing degenerate interactions such as the interaction of a wave and a second harmonic [53], or for specific initial conditions [54].

The equations

$$\dot{a}_{1,2} + V \nabla a_{1,2} = a^*_{2,1} a_3, \quad \dot{a}_3 + V_3 \nabla a_3 = a_1 a_2 \tag{III.9}$$

are the original ones here and have the initial conditions

$$a_j(x, y, z, 0) = a_{j0}(x, y, z). \tag{III.10}$$

It already follows directly from the form of the equations that their solutions have certain similarity properties. Thus, if $a_j(\mathbf{r}, t)$ is the solution of (III.9), (III.10), then $\bar{a}_j = \beta a_j(\beta \mathbf{r}, \beta t)$ is the solution of the same system but with different initial conditions: $\bar{a}_j(\mathbf{r}, 0) = \beta a_{j0}(\beta \mathbf{r})$, where $\beta = \text{const}$. From this there follows specifically the very useful result that the interaction of short pulses having a large amplitude is analogous to the interaction of long pulses having a small amplitude. Problems of the interaction of fast pulses of large amplitude and of slow pulses of small amplitude are similar; if $a_j = a_j(\mathbf{r}, t)$ is the solution of (III.9), (III.10), then $\bar{a}_j = \beta a_j(\mathbf{r}, \beta t)$ is the solution of (III.9), where $V_j \to \beta V_j$, with the initial conditions $\bar{a}_j(\mathbf{r}, 0) = \beta a_{j0}(\mathbf{r})$. For localized fields there follow additionally from (III.9) the integrals

$$N_1 - N_2 = l_1 \quad N_1 - N_3 = l_2, \quad N_2 - N_3 = l_3, \tag{III.11}$$
$$N_j = \int |a_j|^2 dr,$$

which mean that the number of quanta in individual pulses or beams increase or decrease simultaneously (i.e., just as for spatially homogeneous fields,† "explosion" is possible for the interaction of pulses or beams).

As a result of the explosion, the pulses do not have time to disperse, and their increasing fields are concentrated in a narrow needle-shaped region – the merging of pulses of resonance-coupled waves into one pulse takes place. For interaction of beams that propagate at an angle with respect to each other, the merging effect consists in localization of the field coupled waves along a single ray. The merging of pulses for three-wave interaction is illustrated in Fig. 40. These numerical results [53] were obtained for collisions of oppositely traveling pulses having an initial shape in the form of a bell. Here the surprising effect is that the explosion time (its dependence on the initial amplitude is given in Fig. 41) may exceed by an arbitrary amount the time required for linear "hopping" of the pulses past one another (this contradicts our intuition which tells us that for $\Delta V_{gr}/l_{pul} = \tau_{disp} \ll t_{exp}$ each of the pulses must ignore the others).

The merging of pulses takes place only when a certain initial-amplitude threshold is exceeded. For weak initial fields, there is no "explosion," and the pulses disperse. It is easy to estimate the threshold in the stipulated-field approximation for one of the pulses. Then instead of (III.9) we shall have

$$\dot{A}_3 + V_3 A_3' = A_2(x) A_3, \quad \dot{A}_3 + V_3 A_1' = A_2(x) A_1. \tag{III.12}$$

For $t \gg T = l/u$, where $u = V_3 = -V_1$, $A_2(x) = a \cosh^{-2}(x/l)$, this system has the asymptotic solution:

† Besides the case in which wide beams or long pulses interact, spatially homogeneous solutions are also suitable for the case of complete group synchronism: $V_j = V$.

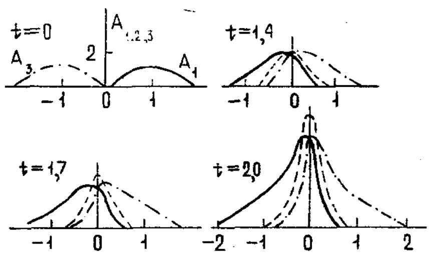


Fig. 40. Merging of pulses for explosion instability.

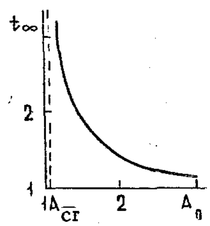


Fig. 41. Dependence of the explosion time on the initial amplitudes of the colliding pulses.

$$A_{1,3} = \frac{C}{\mathrm{ch}^n (x/l)} \exp\left[\mp \frac{x}{2l} + \left(a - \frac{1}{2T}\right) t\right], \tag{III.13}$$

where $n = aT$; c is a constant determined from the initial conditions. It is evident that for $2la/u > 1$ a growth of the fields $A_{1,3}$ takes place which then leads to explosion, while for $2la/u < 1$ the pulses $A_{1,3}(x, 0) \ll A_2(x, 0)$ disperse.

In conclusion, let us turn our attention to one fact which derives from the results presented. The velocity of the leading edges of the interacting pulses during explosive instability is, in practice, in no way related to the linear group velocity of the waves. For example, it may even be substantially greater than the velocity of light. This effect can be explained by the intense generation of all of the waves in the region of the overlap of the pulses. In other words, the velocity of the leading edge of the pulse in a nonequilibrium medium is determined not so much by the transport of quasiparticles from one region of space into another, but by the production of quasiparticles, including in places where there were no such particles. "Faster-than-light" propagation of pulses is also possible for self-action of waves – specifically for propagation of a pulse in a laser medium [55].

IV. WAVE TURBULENCE IN NONEQUILIBRIUM MEDIA

1. General Concepts

For the development of a dynamic beam in the form of nonlinear waves in a nonequilibrium medium where a large number of modes may be excited, one usually requires fairly specific conditions [5] (a resonator, spectrally narrow instability, strong dispersion, etc.). Considerably more frequently a turbulent state develops for excitation of a large number of modes in a nonequilibrium medium – intense wave motion turns out to be disordered. Here let us discuss certain features of turbulent processes in dissipative nonequilibrium media. The mention of the nonequilibrium nature of the medium along with the term "turbulence" possibly appears to be redundant – it is, after all, specifically with the nonequilibrium nature of the medium that the very fact of the development of turbulence is associated.* However, in the majority of cases in which turbulence is discussed, we have in mind only those physical situations for which the instability and dissipation domain may be spread so far apart in k-space that they will play only the role of boundary conditions, while the actual character of the turbulence will depend on the properties of the medium in the inertial interval (where there is no instability or dissipation). We shall understand turbulence in nonequilibrium media to mean turbulence throughout the entire spectral space, including outside the inertial interval, principal attention being devoted to those cases in which the inertial interval is small or absent altogether. It is clear that the properties of such turbulence are not determined by the flow of energy along the spectrum, and this turbulence is not Kolmogorovian. Problems in hydromechanics (turbulence in a boundary layer, the case of thermal convection, etc.), plasma physics (for example, acoustic or Langmuir turbulence in a plasma with collisions), radiophysics (waves of self-excited noise oscillators), and others, lead to the investigation of turbulence of this kind.

In analyzing turbulence, it is usually the case to restrict the analysis to the determination of the turbulence spectrum. At the same time, it is obvious that the spectrum incompletely characterizes the turbulence. For example, in an inertial interval the Kolmogorovian spectrum $k^{-5/3}$ is also obtained for a strong vortical turbulence and a weak wave turbulence within the framework of the random-phase approximation (for example, for gravitational-capillary waves [56]), as well as for other models. Moreover, the spectrum has nothing of the mechanism governing the development and establishment of the turbulence or of its structure, whose investigation is of obvious interest. It is specifically to these problems that we shall devote the concluding portion of the lectures.

In both wave and hydrodynamic turbulence, the most frequently encountered (and therefore already fairly familiar now) mechanism for the development of turbulence is associated with the development of a chain of successive instabilities and the relay transfer of energy from some scales to others [16] (more frequently from large ones to small ones). It is specifically this mechanism that corresponds to Kolmogorovian turbulence. However, for nonequilibrium media it is more likely an exception. In nonequilibrium dissipative

*In equilibrium media, it is also possible for disordered motions to occur in which many degrees of freedom are occupied, but these motions correspond not to turbulence but to thermalization – a redistribution of energy among various modes which leads to the establishment of thermodynamic equilibrium. The intensity of these motions is determined uniquely by the temperature of the medium, which is the only parameter of the steady-state Gibbs distribution.

media, the mechanism for the development of turbulence turns out to be very specific. This specific mechanism is caused by two factors: 1) due to linear especially nonlinear instability (for example, explosion instability) it is possible to have a simultaneous growth of perturbations having completely different scales, whose amplitude limitation may be caused both by the transfer of energy to attenuating modes and by nonlinear dissipation of energy within the instability domains, and 2) for multimode interactions in nonequilibrium dissipative media, the absence of randomization of the phases of the modes is most characteristic (i.e., essentially, weak turbulence is possible). In view of the importance of the latter fact, let us dwell on it in greater detail.

Using the example of three-wave processes, we have seen (see Para. 4, Sec. II) that even in media having a small nonlinearity the phases either synchronize or vary with the same characteristic times as the amplitudes for interacting of unstable modes with each other or with attenuating modes. As the physical and computer experiments (see below) showed, the situation is also the same for multimode processes. Qualitatively, the difference from "equilibrium" media may be explained here as follows: For a sufficiently large width of the inertial interval, the phase which is imposed on the harmonics lying in the transparency domain by the unstable modes is forgotten with departure from the boundary of the instability domain due to interaction with a large number of modes within the inertial interval (it is assumed that the medium has a weak dispersion). However, if the inertial interval is absent, the attenuation that progresses with a growth of the mode number breaks this relay, and the unstable modes efficiently excite only several attenuating modes which are close to exact resonance along with them. Such a picture is also observed in media having an inertial interval that is not wide, when the number of interactions within the inertial interval is not too high. As an example, we present the results of a computer experiment [57] on the stabilization of an unstable ion-sound mode due to the transfer of its energy upward along the spectrum. A model consisting of 10 modes with a growth rate γ_0 for the first mode and an attenuation $\nu = 10\gamma_0$ for the tenth mode was investigated (the inertial interval was nine harmonics). It turned out that both for small dispersion and in the absence of dispersion the cascade transfer of energy from mode to mode does not occur, and the stationary energy distribution $E(\omega)$ corresponding to a constant energy flow along the spectrum is not established. The energy of all of the modes oscillates, varying by more than two orders of magnitude, it being true that the energy "bursts" for various modes take place with a small time spread (almost in-phase) – see Fig. 42.

Thus, even for a small nonlinearity the approximation involving random phases and a weak coupling between modes (small energy exchange between them) is usually invalid in nonequilibrium dissipative media, and in this sense the turbulence of such media is usually strong.* Let us consider the principal mechanism for the development and establishment of such a turbulence by relating them to the character of instabilities in the media.

2. Turbulence in Media Having Linear Instability and Attenuation

Let us investigate how a low-frequency (acoustic) turbulence is excited, for example, as a result of parametric instabilities in a nonlinear dissipative medium having dispersion. In order to be specific, we shall assume that in the linear approximation the spectrum of the low-frequency waves is analogous to the spectrum of ion sound in a nonisothermal plasma (a similar spectrum characterizes helicons, spin waves, etc.),

*Note that strong wave turbulence has already been discussed frequently both for dissipative media (Bürgers turbulence – an ensemble of sawtooth waves with random phases [58]) and for media close to conservative media (for example, Langmuir turbulence in the form of a gas of solitons [59]).

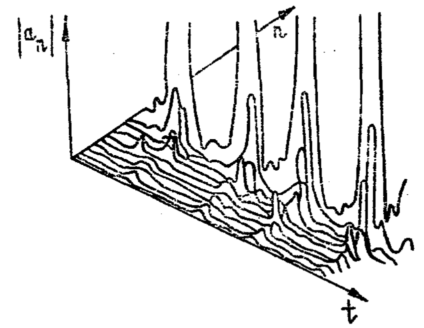


Fig. 42. Oscillations of the energy of the harmonics for instability associated with the first harmonic and attenuation associated with the tenth harmonic [57].

while the mode which flows due to parametric instability of waves of a different type (for example, Langmuir waves) lies on the boundary of the strong-dispersion domain. Under these conditions, energy transfer upward along the spectrum does not take place, and one may limit the analysis solely to decay processes.

a) Narrow Excitation Spectrum. The original equations for complex amplitudes of the spectral components which interact in a quadratic medium can be written in the form

$$\dot{b} = -\beta\omega_0 \int_{\Delta}^{k_0/2} a_k a_{k_0-k} e^{i\Delta\omega(k)t} dk + \gamma b; \tag{IV.1}$$

$$\dot{a}_k = \beta\omega(k) b a^*_{k_0-k} e^{i\Delta\omega(k)t} - \nu_k a_k. \tag{IV.2}$$

Here b is the complex amplitude of the wave at the frequency ω_0 that is parametrically excited with the growth rate γ; a_k is the complex amplitude of the k-th component of the spectrum; ν_k is its attenuation rate; $k = k(\omega)$, $k_0 = k(\omega_0)$. As follows from (IV.2), the stabilization of an unstable mode due to decay and energy dissipation in modes having longer wavelengths is possible only in the presence of fluctuations in the lower portion of the spectrum. If the level of the initial fluctuation does not depend on frequency, then those spectral pairs that are symmetrical relative to $\omega_0/2$, $k_0/2$ will be predominantly excited for which $k \sim k_0/2$ – for them the decay growth rate is a maximum; however, pairs having $|(k_0/2) - k| \sim k_0/2$ that are far from the center of the spectrum do not grow at all, which is taken into account in (IV.1) $(2\Delta/k_0 \ll 1)$.

To answer the question of the nonlinear evolution of the spectrum of initial fluctuations, we make use of the results of a computer experiment on the interaction of different decay pairs [34]. It turns out that close pairs are mutually synchronized with respect to their phases and form coupled states – domains in k-space.† In the dynamic case (there is no dispersion: $\Delta\omega = 0$), the spectral width Ω of the k-domain decreases slowly with a growth of the relative attenuation of the pairs – for $\nu/\gamma = 10$, $\Omega = 0.1$, while for $\nu/\gamma = 30$, $\Omega = 0.06$. When dispersion is taken into account – stochastic triplets (see Para. 4, Sec. II) – the width of the k-domain is independent of ν/γ over wide limits and increases linearly with a growth of the detuning $\Delta\omega$ within the triplet. For example, for $10 < \nu/\gamma < 100$, $\Omega \approx 30\Delta\omega$.

For a spectrally narrow growth rate of the unstable mode, only one pair of domains may exist at each time, as experiment has shown. This is evidently related to the fact that different pairs are supplied from one spectrally narrow source and suppress each other during interaction.‡ The presence of fluctuations leads to a situation in which the lifetime of an arbitrary pair of domains turns out to be finite, and the system goes over randomly from a state with one excited pair to a state with the other excited pair. Physically, this result appears to be fairly obvious since decay pairs actually exist only during a small fraction of the "period" τ_0 of the triplet (their intensities have the form of sequences of narrow peaks; see Fig. 32); the random intensity spikes of the other spectral components in the interval between peaks induce the decay of the unstable mode into a new pair, etc. In the computer experiment, the lifetime of the pair turned out to equal $\tau_{pair} \sim (10\text{–}50)\,\tau_0$. The frequency distribution function of the pairs had a maximum close to $\omega_0/2$ (Fig. 43).

Thus, for a narrow excitation spectrum the turbulence in the case given represents an ensemble of decay pairs of k-domains that "girdle" in time – at each time only one pair exists that is replaced by another according to a random law. In view of the fact that the frequency of an appearance of the pairs increases as

† The process of concentrating the energy of a decaying mode in domains is reminiscent of the development of jets in k-space – for example, for scattering of waves by particles [60]. The distinction of these processes resides specifically in the fact that the phases of different modes within the jet are random, while inside a domain the phase of the spectral components is constant and is not dependent on ω (or on k).

‡ This process is similar to mode competition in a laser with a homogeneously broadened line of an active substance [61].

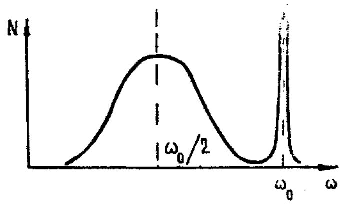


Fig. 43. Stationary turbulence of a "gurgling" triplet.

$\omega_0/2$, the spectrum of stationary turbulence must qualitatively appear as it does in Fig. 47 in the case ν_k = const.

b) Kinetic Equation for Decay Triplets. If the instability domain of decaying modes is wide in ω-space in comparison with the width of the domains, it is already possible for different pairs of domains to coexist.* In this case, the turbulence is an ensemble of decay triplets whose properties we already know. The spectrum of this turbulence in the zero approximation ("ideal" triplet gas) is simply a superposition of the spectra of individual triplets which are distinguished by the frequencies of the hf modes and their growth rates.

In accordance with computer experiments, decay triplets turn out to be stable relative to finite external perturbations. Therefore, when collisions are taken into account in a gas of triplets, the latter may be treated as complex indivisible quasiparticles. One should describe such a gas by the kinetic equation for these quasiparticles. Unlike the kinetic equation for elementary quasiparticles (i.e., quasiharmonic waves whose phases are assumed to be random), in the case given the kinetic equation is derived naturally without resort to additional hypotheses of the phf type [39]. This can be explained by the fact that the phases of the waves in each triplet jump due to internal interactions, and in analyzing processes in which waves from different triplets participate the phases should be considered random.

The equations describing a "real" gas of triplets are not difficult to obtain if the following is taken into account. The interaction of decay triplets is realized only due to long-lived unstable modes (see Fig. 21), since attenuating modes from different triplets do not overlap in time – they have the form of short pulses whose duration is substantially shorter than their average sufficient time, the bursts of lf modes from different triplets being uncorrelated. However, the interaction between unstable hf modes may be caused only by those sound perturbations whose frequencies and wave numbers are smaller than or of the order of the spectral widths of the instability domain.† Since in our case the width of the instability domain is assumed to be much smaller than the frequency of the hf modes, the desired equations are found within the framework of the adiabatic approximation [62] in which hf modes having random phases propagate in a slowly varying (due to low-frequency sound) medium. As a result, we shall have the kinetic equation for the hf modes and the hydrodynamic equation for the lf sound for parametrically excited one-dimensional sound waves:

$$\begin{aligned} &\frac{\partial N_k}{\partial t}+v_k\frac{\partial N_k}{\partial x}-s\frac{\partial^2\xi}{\partial x^2}\frac{\partial N_k}{\partial k}=\gamma_k N_k-\rho_k N_k^2,\\ &\frac{\partial^2\xi}{\partial t^2}-v_s^2\frac{\partial^2\xi}{\partial x^2}=-s\frac{\partial}{\partial x}\sum_k N_k+\nu_s\frac{\partial^3\xi}{\partial t\,\partial x^2}. \end{aligned} \tag{IV.3}$$

Here N_k is the intensity of the k-th hf mode; ξ is a variable characterizing the low-frequency sound perturbation; s is a constant which depends on the normalization; ν_s is the viscosity; γ_k is the linear growth rate of the hf modes, while $\rho_k N_k^2$ is a model term describing the stabilization of the hf mode due to decay into attenuating modes inside the triplet, it being true that $\gamma_k/\rho_k = N_k^{(0)}$ (the average level of the hf mode in the stationary-stabilization regime in an autonomous triplet).

An "ideal" triplet gas corresponds to a state having $\xi = 0$, which exists only for a uniform distribution of the energy of the hf modes in space. It can easily be seen that such a state is unstable relative to longwave perturbations. Specifically, the stationary regime $N_k(x, t) = N_k^{(0)}$ is unstable relative to perturbations having the characteristic scale

$$\begin{aligned} &\Lambda > 2\pi\left[\frac{v_k\nu(v_k v_s^2-\alpha s^2)}{\gamma v_s^2(v_s^2-\nu\gamma)}\right]^{1/2} \qquad \left(\alpha<\frac{v_k v_s^2}{s^2}=\alpha_{cr}\right),\\ &\Lambda > 0 \qquad\qquad\qquad\qquad\qquad\qquad (\alpha>\alpha_{cr}). \end{aligned} \tag{IV.4}$$

Here $\alpha = N_k^{(0)}(k_2) - N_k^{(0)}(k_1)$; $k_{1,2}$ are the boundaries of the domain in k-space where the growth rate of the hf modes is nonzero.

*Continuing the analogy with lasers, it may be noted that this case is similar to simultaneous generation of many modes in a medium with inhomogeneous broadening of a line – different modes de-excite different active particles.

† Let us recall that we are considering waves whose spectra is analogous to the spectrum of ion-sound – at low frequencies there is no dispersion, while at high frequencies the dispersion is strong, as a consequence of which the merging processes are forbidden.

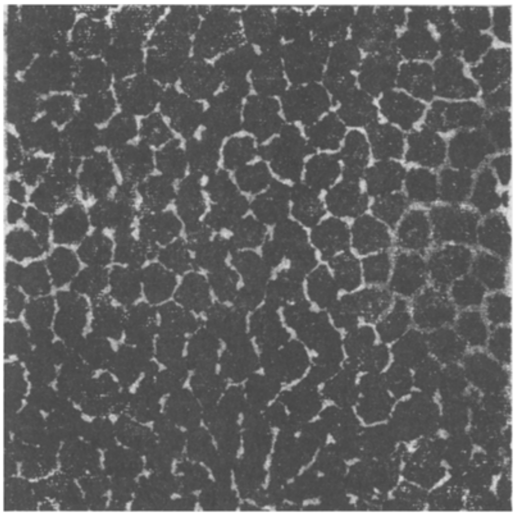


Fig. 44. Conservation of the spatial structures associated with explosive interaction in turbulent thermal convection [69].

Thus, the gas of triplets is unstable relative to the partitioning of a spatially uniform distribution into bunches.

3. Turbulence Structure for Explosive Instability

Resonance interaction of a large number of modes in a nonequilibrium medium for explosive instability can be described by an equation of the following form in a fairly general case:

$$\dot{a}_k = \gamma_k a_k + \sum_{i,j} \sigma_{ijk} a_i a_j - \sum_{l} \rho_{lk} a_k \mid a_l \mid^2. \qquad \text{(IV.5)}$$

Here γ_k is the linear growth rate or attenuation rate; σ characterizes the nonlinear growth rate (all σ_{ijk} are real and of the same sign, which ensures explosive instability); the last term describes the nonlinear attenuation which is associated either with the transfer of energy over the spectrum by means of rapidly attenuating modes (they are expressed algebraically in terms of a_k) or with a nonlinear viscosity or conductivity. This equation describes the processes of interaction of Tollmien – Schlichting waves in a boundary layer with drainage, "explosion" modes in thermal convection in a layer that is heated from the bottom by a liquid whose viscosity depends on temperature [63], waves in an active waveguide [68], etc.

The character of the turbulence which develops as a result of the excitation of a large number of modes is determined in the case given by the relationship between the parameters γ, σ, and ρ. For $\sigma^2/\rho\gamma \ll 1$, the steady-state intensity of the turbulent pulsations is of the order of $N_0 \sim \gamma/\rho$, and the explosive interaction takes effect only during the last stage when the nonlinear stabilization and nonlinear instability processes approximately balance each other. The explosion phase-synchronization time turns out here to be substantially greater than the time for interaction due to cubic nonlinearity. Therefore, for $\sigma^2/\rho\gamma \ll 1$ the process may be described by a conventional kinetic equation derived in the approximation based on random phases of the waves (weak turbulence):

$$\dot{N}_k = 2\gamma_k N_k - 2 \sum \rho_{lk} N_l N_k + \sum |\sigma|^2 (N_i N_k + N_j N_k + N_i N_j). \qquad \text{(IV.6)}$$

The explosive interaction described in (IV.6) by the last term leads only to a redistribution of the energy along the modes. Thus, in the case given the "explosion" essentially has no effect on the structure of the turbulence.

The situation will be qualitatively different if the "explosion" phase-synchronization time $t_\infty \sim 1/\sigma A$ turns out to be less than $\tau_{\text{nonlin}} \sim 1/\rho A^2$. In this case, well-defined formations – explosion triplets – will develop in the beginning from the fluctuations as a result of the action of explosive instability; these formations, after they have formed, will interact with one another. Under these conditions, they will remain internally stable – the amplitudes of the modes inside the triplet will be uniquely coupled, while the phases will be mutually synchronized. Repeating the reasoning that is usually used as the basis for the random-phase approximation for

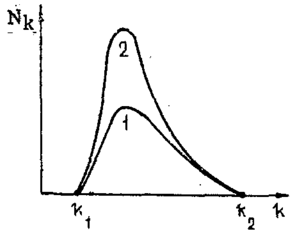


Fig. 45. Dependence of the turbulence spectrum accompanying thermal conduction in a liquid having $\nu = \nu(T°)$ on the turbulence structure: 1) spectrum of weak turbulence; 2) spectrum of turbulence in the form of an ensemble of explosion triplets.

simple quasiparticles, one may consider the interaction of explosion triplets with noncommensurate scales (k) within the framework of the kinetic equation for triplets* [63]

$$\dot{N}_k = 2\gamma_k N_k + \sigma N_k^{3/2} - 2\sum_i \rho_{ik} N_i N_k + \sum |\sigma|^2 (N_i N_k + N_i N_j + N_j N_k), \tag{IV.7}$$

where N_k is the intensity of a triplet for which all three modes have a wave number of the same order of magnitude (in the majority of cases it is precisely for them that the "explosion" growth rate σ turns out to be maximal).

Turbulence in the form of an ensemble of triplets was evidently observed experimentally in a layer of heated liquid in which the viscosity depended on temperature [69] – see Fig. 44.

Thus, for $\sigma^2/\rho\gamma \sim 1$ the turbulence in a medium having explosion instability is no longer weak and constitutes an ensemble of interacting explosion triplets. A change in the turbulence structure also leads to a substantial change in its spectrum. In [63], the turbulence spectra are found in the case when the Rayleigh number is of the order of several critical numbers for thermal convection in a layer of liquid heated from the bottom. It turned out that the development of triplets leads to an increase in the energy of turbulent pulsations in the region of large scale – see Fig. 45.

4. Multiphase Turbulence

The turbulence in the form of an ensemble of triplets (explosion or decay) investigated earlier is a particular case of strong wave turbulence. Bürgers turbulence in the form of an ensemble of sawtooth waves having random phases [58] and gas solitons [70] should also be classified as such turbulences. In the language of quasiparticles, strong turbulence is an ensemble of complex quasiparticles having strong internal stability and interacting weakly with each other.

However, it can easily be noticed that turbulence in the form of a gas of identical quasiparticles exists only under fully defined (and sometimes fairly specific) conditions. Thus, in order for acoustic turbulence to be of the Bürgers type it is necessary for there to be an almost complete absence of dispersion, and for the turbulence to be weak the opposite is true – the dispersion must be large. In a practical situation, these two cases simply correspond to the region of small wave numbers (Bürgers turbulence with an energy spectrum $E_k \sim k^2$ [64]) and to the region of large wave numbers (weak turbulence having the spectrum $E_k \sim k^{-2/3}$ [67]). Thus, even from this example it follows that in the general case wave turbulence must be "multiphase" turbulence (i.e., it is a mixture of gases – phases consisting of different quasiparticles). The interaction between these "phases" is most substantial in the region of transitional scales in k-space. Such multiphase models were considered with application to sound [65] and Langmuir [66] turbulences.†

Let us now discuss the multiphase turbulence that develops for interaction of low-frequency and high-frequency waves (for example, ion-sound and Langmuir waves in a plasma [66] or internal and surface waves in the ocean) in greater detail. The interaction of one-dimensional waves can be described by the system

$$\begin{aligned} -2ia_t + \beta a_{xx} - \omega_p na &= -2i\nu_1 a_{xx}, \\ n_{tt} - c_s^2 n_{xx} &= \gamma |a|^2_{tt} + \delta n_{xxxx} + \nu_2 n_{xxt}, \end{aligned} \tag{IV.8}$$

where β characterizes the dispersion of the hf waves; ω_p is their critical frequency (see the dispersion characteristic in Fig. 46); δ characterizes the dispersion of the lf waves; c_s characterizes their velocity;

*Unlike decay triplets, the phases of modes inside an explosion triplet do not "jump" – the explosion triplet does not have internal degrees of freedom. Therefore, the derivation of the kinetic equation is based on the approximation of the randomness of the phases of the triplet.

†The model for turbulence in the form of a mixture of gases "saws" and conventional quasiparticles proposed in [65] allowed splicing of the spectra for Bürgers and weak turbulences.

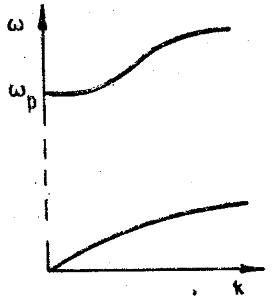


Fig. 46. Dispersion characteristic of hf and lf normal waves in an LC line.

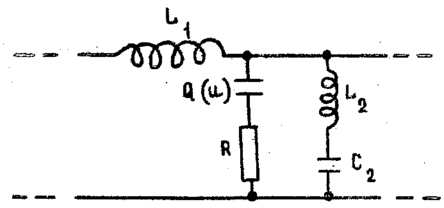


Fig. 47. Equivalent circuit of a cell.

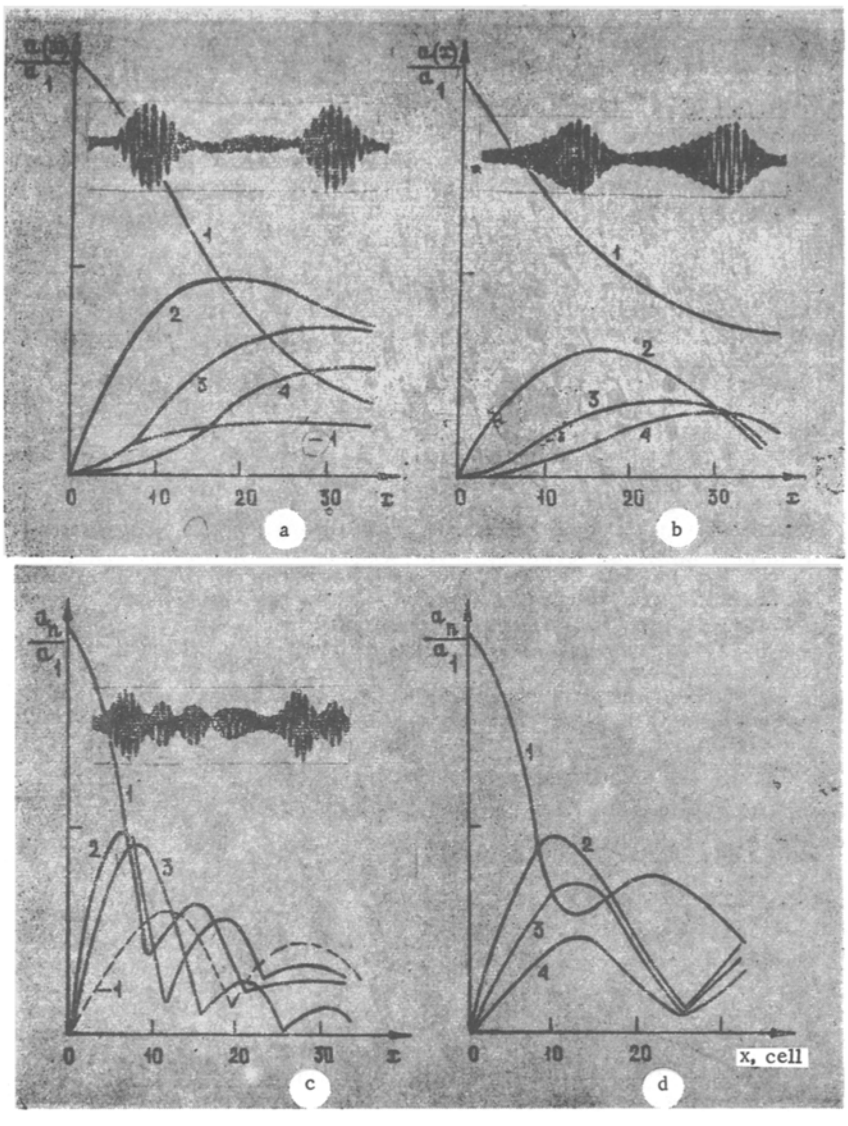


Fig. 48. Oscillogram and evolution of the spectra of the envelope waves: a) soliton ($\omega/\omega_p = 1.08$, $\Delta\omega/\omega_p = 0.05$); b) shock waves ($\omega/\omega_p = 1.04$, $\Delta\omega/\omega_p = 0.05$); c) dynamic turbulence ($\omega/\omega_p = 1.36$, $\Delta\omega/\omega_p = 0.05$); d) periodic energy exchange ($\omega/\omega_p = 1.13$, $\Delta\omega/\omega_p = 0.15$).

ν_1 and ν_2 are the viscosity of the hf and lf waves (for a plasma, ν_1 simulates the Landau damping of Langmuir waves); γ characterizes the magnitude of the coupling between hf and lf waves. The system (IV.8) is usually solved numerically. We shall make use of its solution obtained as a result of physics experiments on the interaction of hf and lf waves in a one-dimensional nonlinear medium – a line with nonlinear capacitances (the equivalent circuit of a section is displayed in Fig. 47). A semiinfinite "medium" was investigated on whose boundary monochromatic hf and lf waves were excited. For an identical energy of the hf wave on the boundary, four qualitatively different forms of hf turbulence develop as a function of its frequency (see Fig. 48): solitons, sawtooth waves, dynamic turbulence – an ensemble of hf waves that quasiperiodically exchange energy with each other, and a regime with periodic exchange of energy between satellites which develop as a result of modulation instability and the "carrier" wave which is intense on the boundary. In the experiments that were performed, several tens of satellites were observed; the first four or five of them are shown in Fig. 48 as a function of x.

Envelope solitons developed due to modulational instability in the region of strong dispersion ($\omega \gg \omega_p$) for low dissipation. Figure 48a displays an oscillogram of envelope waves in the form of a sequence of solitons and shows the evolution of their spectrum.

For a reduction of the frequency of the lf wave incident on the boundary ($\omega \to \omega_p$), the solitons were replaced by waves having a sawtooth shape (see Fig. 48b).* If hf waves having a frequency lying within that region of the dispersion curve where group synchronism with lf waves is satisfied were excited on the boundary, then a dynamic-turbulence regime was established (Fig. 48b). In the absence of harmonics of the lf waves, this regime went over into a rigorously periodic regime – in Fig. 48d it is evident that the satellites simultaneously return the energy to the initially intense wave and are then intensified by this wave again.

For excitation of hf waves with a wide spectrum on the boundary of the medium, a multiphase-turbulence regime develops which consisted of the result of the interaction of the elementary "gases" considered above. The presence of complex quasiparticles of each kind in such turbulence was identified according to the spectra. An analysis of the pair interaction of different "phases" demonstrated that quasistationary elementary regimes were established relatively rapidly and then slowly exchanged energy with each other.

In concluding the present section, we note that, regrettably, in practice all results involving strong wave turbulence were obtained either by means of numerical or by means of model experiments. This is associated both with the colossal analytic difficulties of the problems considered and with the fact that their intensive investigation is essentially only beginning.

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WAVE – PARTICLE INTERACTION IN NONEQUILIBRIUM MEDIA

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UDC 538.57

1. INTRODUCTION

The first part of the present course of lectures was devoted to an investigation of nonlinear effects which are based on the wave – wave interaction. An important feature of plasma turbulence, which distinguishes it from hydrodynamic turbulence, resides in the fact that in a plasma the interaction of waves with resonance particles whose velocities $\mathbf{v}$ are related to the frequency ω_k and the wave vector $\mathbf{k}$ of the wave by the condition

$$\omega_k = \mathbf{kv} \text{ (plasma with no magnetic field)},$$
$$\omega_k = k_\| v_\| + n\omega_H \text{ (magnetically active plasma; } n = 0, \pm 1, \ldots), \tag{1}$$

where $\omega_H = eH_0/mc$ is the cyclotron frequency, plays an essential and sometimes even a dominant role along with the effects resulting from wave – wave interaction. In a nonequilibrium Maxwellian plasma, resonance wave – particle interaction leads to the development of collisionless wave attenuation – Landau damping. For a deviation of the distribution function of the resonance particles from an equilibrium distribution (a beam in a plasma, temperature anisotropy), the same mechanism of the interaction of waves with particles leads to the development of an extensive group of microinstabilities.

As a result of interaction with waves, the distribution function in the region of resonance velocities is deformed in such a way that the exchange of energy between particles and waves ceases (collisionless relaxation of the distribution function). Two approaches are possible through an investigation of this process. One of them, which has been developed in the greatest detail, is based on the use of the equations of quasilinear theory [1] and is applicable when a broad wave packet in which the phase "mixing" time $t \sim 1/\Delta k|v - v_{gr}|$ (Δk is the width of the packet; v_{gr} is the group velocity) is substantially shorter than the quasilinear relaxation time of the distribution of the resonance particles interacting with the plasma. For Langmuir oscillations, this condition means that the width of the wave packet according to the phase velocity substantially exceeds the velocity with which the particles oscillate in the potential well created by the packet:

$$\Delta\left(\frac{\omega_k}{k}\right) \gg \sqrt{\frac{e\varphi_0}{m}}, \qquad \varphi_0 = \sqrt{\sum_k |\varphi_k|^2}. \tag{2}$$

Along with (2), it is necessary in quasilinear theory for the condition ensuring "collectivization" of the motion of resonance particles in the packet to be fulfilled – the phase-velocity distance between the individual harmonics of the packet must be substantially smaller than the width of the potential well for these harmonics:

$$\delta\left(\frac{\omega_k}{k}\right) \ll \sqrt{\frac{e\varphi'}{m}} \qquad \left(\varphi'^2 \approx |\varphi_k|^2 \delta k, \ \frac{\delta k}{k} \sim \frac{k}{\omega}\sqrt{\frac{e\varphi'}{m}}\right), \tag{3}$$

so that condition (3) may likewise be written in terms of the spectral density of the potential in the form

$$\delta\left(\frac{\omega_k}{k}\right) \ll \left(\frac{e^2}{m^2 v} k|\varphi_k|^2\right)^{1/3}. \tag{3'}$$

Under these conditions, the change in the distribution of the resonance particles due to an individual harmonic of the packet is slight, while the joint action of many harmonics of the spectrum leads to a slow velocity diffusion of the resonance particles which ceases when a "plateau" is formed on the distribution function. The

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