SUPERREFLECTION OF WAVES IN HYDRODYNAMIC FLOWS

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The amplification mechanism of various type waves in hydrodynamics is analyzed for reflection from planar and cylindrical tangential discontinuities. The problem of wave momentum and energy in a medium is discussed. The amplification is related to the presence of negative energy waves.

1. INTRODUCTION. WAVE ENERGY AND MOMENTUM IN A MEDIUM

The study of various type wave propagation in hydrodynamic flows requires, firstly, the explanation of physical mechanisms of the "wave-flow" interaction: amplification, absorption, and scattering of waves in hydrodynamics. These problems refer not only to hydrodynamic systems; they are also generated, in particular, in the study of electromagnetic effects, in electrodynamics of continuous media [1], in plasma theory [2], and are, essentially, general problems of physics of nonequilibrium media. Qualitative understanding of the interaction mechanisms of waves with a moving medium is necessary for developing instability theories [3, 4], the study of nonlinear effects in nonequilibrium media [5], etc.

In the present study we consider various aspects of one of the most effective mechanisms of wave amplification and absorption in a nonuniformly moving medium, related to the presence of negative energy waves and the change in sign of dissipation in a hydrodynamic flow. Unlike the well-known resonance interaction mechanism of waves with synchronous frequencies in the critical layers [4, 6], here the whole flow participates in the interaction, and the effect is independent of the details of the velocity distribution in the flow. Therefore, the features of this mechanism are conveniently investigated on the example of simplest hydrodynamic flows: tangential discontinuity (TD) and other flows with piecewise constant vorticity, where the effect occurs in pure form and admits analytic study.

The law of wave energy conservation is widely used in flows for the interpretation of wave theory results. The concept of wave energy of a process in a continuous medium is far from trivial. Linearizing the original equations of motion, one can obtain conservation laws related to the steady state and homogeneity of the unperturbed medium. The values of ε and $\boldsymbol{P}$, conserved for a monochromatic wave $\exp(-i\omega t + i\boldsymbol{kr})$, and expressed in terms of the wave action Q [7] (the analog of the quasiparticle number in quantum field theory): $\boldsymbol{P} = \boldsymbol{k}Q$, are usually called the wave energy and momentum densities. However, considering the energy and momentum densities in their primary sense (conserving quantities related to the independence of the laws of motion of time and place), we must average the full expressions for energy and momentum, following from the original (nonlinear) system of equations over the wave phase, extracting the quadratic part in the amplitude due to the wave. In this case the quadratic terms, which are neglected in linearizing the original system, can, generally speaking, provide a contribution comparable to the quantities ε and $\boldsymbol{P}$. The motions corresponding to these terms have the meaning of induced wave flows. The problem of energy and momentum of waves, with account of mean motions induced by waves, was discussed multiple times in acoustics, hydrodynamics [8], and electrodynamics of continuous media [1, 9].

If the unperturbed medium is in rest, then the induced wave flow, whose velocity is quadratic in amplitude, obviously provides no contribution to the energy. As to the momentum of wave motion, it can be conveniently separated into the quantity $\boldsymbol{P}$, obtained within the linearized approximation ("pseudomomentum" [10]), and the momentum of the induced wave flow. The characteristic features of these two components can be verified on the example of gravity waves on the surface of deep water.

Linearizing the two-dimensional boundary-value problem for the potential and the surface response η [11],

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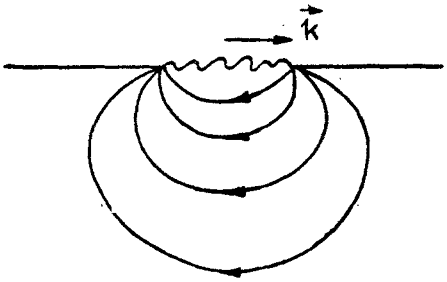

Fig. 1. Mean induced flow of a train of surface gravity waves.

$$\Delta\varphi = 0, \quad \begin{cases} \dot{\eta} + \dfrac{\partial\varphi}{\partial x}\dfrac{\partial\eta}{\partial x} = \dfrac{\partial\varphi}{\partial y} \\ \dot{\varphi} + g\eta + \dfrac{(\nabla\varphi)^2}{2} = 0 \end{cases} \quad \text{for} \quad y=\eta, \tag{1}$$

one easily obtains a solution in the form $\varphi=\varphi_0 \exp(-i\omega t + ikx + |k|y)$ under the condition $\omega^2 = g|k|$. If there is no induced mean flow ($\langle V\rangle = 0$), then the mean horizontal Lagrange particle velocity (the Stokes drift) is

$$\langle u_L \rangle = \langle u\rangle + \langle(\xi\nabla)\, u\rangle = \left\langle \xi\frac{\partial u}{\partial x} + \zeta\frac{\partial u}{\partial y}\right\rangle = \frac{k}{\omega}\langle u^2 + v^2\rangle\,, \tag{2}$$

where $V = (u, v) = \nabla\varphi$, and $\xi = (\xi, \zeta)$ is the displacement of medium particles. The mean Lagrangian particle velocity decreases with depth as $\sim \exp(2|k|y)$, and determines the horizontal mass transport:

$$S_m = \rho\int_{-\infty}^{0}\langle u_L\rangle\, dy = \frac{\rho}{2|k|}\langle u_L(0)\rangle = \rho\langle u(0)\,\eta\rangle. \tag{3}$$

The same mass flow can also be obtained in the Euler description, taking into account mass transfer between peaks and troughs of waves: $S_m = \langle\rho\int_{-\infty}^{\eta} u dy\rangle = \rho\langle u(0)\eta\rangle$.

The Stokes drift is related uniquely to the wave momentum density:

$$P = \int_{-\infty}^{0}\langle\rho u_L\rangle dy \equiv S_m = \frac{k}{\omega}\int_{-\infty}^{0}\rho\langle u^2 + v^2\rangle dy = \frac{k}{\omega}\varepsilon. \tag{4}$$

For the mean flow, slowly varying within the scale of wave oscillations, within second order in amplitude one can obtain from (1) the boundary-value problem [12]

$$\Delta\Phi = 0, \quad \dot{h} - \frac{\partial\Phi}{\partial y} = -\frac{\partial}{\partial x}\left(\eta\,\frac{\partial\varphi}{\partial x}\right), \quad \dot{\Phi} + gh = 0\,, \tag{5}$$

where Φ and h are the mean potential and the surface response. Taking into account that the wave group velocity is small in comparison with the phase velocity of long-wave perturbations, having the scale of the mean flow, we can neglect time derivatives. As a result we obtain the quasistatic problem of flow under the moving distribution of mass sources on the surface y = 0. Obviously, the mass source is the gradient of mass flow S_m, related to the Stokes drift. [For a packet of surface waves the pattern of line current is generated, which is easily found in this approximation, using the analogy with electrostatics: It coincides with the pattern of force lines of charges on the surface y = 0 (see Fig. 1)].

It is easily shown that in each cross section $x = x_0$ the wave momentum P, related to the Stokes drift, is exactly compensated by the momentum of the induced wave flow. Indeed, from (5) we have in the quasistatic approximation

$$\frac{\partial}{\partial x}\left[\rho\int_{-\infty}^{0}\frac{\partial\Phi}{\partial x}\, dy + \rho\left\langle\eta\,\frac{\partial\varphi(0)}{\partial x}\right\rangle\right] = 0\,. \tag{6}$$

The first term in the square brackets provides the mass flux in the induced flow through the cross section x = const, and the second term provides the mass flux of the Stokes drift.

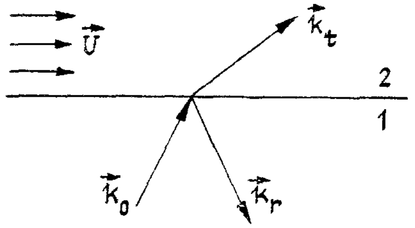

Fig. 2. Wave reflection from a tangential discontinuity. $\boldsymbol{k}_0$, $\boldsymbol{k}_r$ and $\boldsymbol{k}_t$ are the wave vectors of the incident, reflected, and transmitted waves.

A similar study was also performed for other types of waves [10]. It must be stressed, however, that for processes of wave emission, absorption, and scattering, wave propagation in inhomogeneous media and other processes, for whose description the linear approximation is suitable (and only these problems are considered below), the induced wave flow, being a nonlinear effect, can simply be ignored, assuming that the wave momentum and energy in the resting medium are related by $\boldsymbol{P} = (\boldsymbol{k}/\omega)\varepsilon$. In a moving medium, then, using the Laplace transformation, we obtain

$$\varepsilon = \varepsilon_0 + PU = \varepsilon_0 \frac{\omega}{\omega - kU}, \tag{7}$$

where $\mathbf{U}$ is the velocity of motion of the medium, and ε_0 is the energy density in the reference system moving with the medium.

It is precisely in this sense that we perceive the wave energy, considering a wave with negative energy. This description, using, in fact, the concept of wave action and the pseudoenergy and pseudomomentum introduced in [10], is closed and compatible within the linear theory. In this case the conservation of true energy and momentum can be generated by nonlinear emission effects of long-wave perturbations [10, 12] (with a wave train scale), which must be the subject of a separate study.

2. SUPERREFLECTION

2.1. Miles–Ribner Problem. Sound in a Moving Medium

The tangential discontinuity (TD) is the simplest hydrodynamic flow capable of amplifying waves reflected from it (Fig. 2). This effect (superreflection) was first noted for sound incident on a TD [13, 14], and was then also treated for other types of waves: internal gravity [15], electromagnetic [16], etc.

We discuss in detail the simple problem of reflection of a monochromatic wave $\exp(-i\omega t + ikx)$ from a TD. Matching the solutions for the potential, the pressure p, and the nontrivial particle displacement ζ (in the y direction) in resting (1) and moving (2) media,

$$\begin{gathered}\varphi_1 = e^{iq_1 y} + Re^{-iq_1 y}, \quad p_1 = i\omega\rho\varphi_1, \quad \zeta_1 = -\frac{q_1}{\omega}\left(e^{iq_1 y} - Re^{-iq_1 y}\right), \\ \varphi_2 = Te^{iq_2 y}, \quad p_2 = i(\omega - kU)\rho\varphi_2, \quad \zeta_2 = -\frac{q_2}{\omega - kU} Te^{+iq_2 y},\end{gathered} \tag{8}$$

by means of the boundary conditions $[p_1 - p_2]_{y=0} = 0$, $[\zeta_1 - \zeta_2]_{y=0} = 0$ we find the reflection and transmission coefficients

$$R = \frac{q_1/\omega^2 - q_2/(\omega - kU)^2}{q_1/\omega^2 + q_2/(\omega - kU)^2}, \quad T = \frac{2q_1/[\omega(\omega - kU)]}{q_1/\omega^2 + q_2/(\omega - kU)^2}, \tag{9}$$

where $q_1 = [\omega^2/c^2 - k^2]^{1/2}$, $q_2 = [(\omega - kU)^2/c^2 - k^2]^{1/2}$, and c is the speed of sound. The sign of the vertical component q_2 of the wave vector in a moving medium is determined by the emission condition $v_{gr\,y} > 0$, which can be obtained by solving the initial problem [17]. It follows from the dispersion equation $(\omega - kU)^2 = c^2(k^2 + q_2^2)$ that: $v_{gr\,y} \equiv \partial\omega/\partial q_2 = c^2 q_2/(\omega - kU)$. For $\omega - kU < 0$ the radiation condition requires to select the branch $q_2 < 0$. In this case the reflected wave is amplified: $|R| > 1$.

The interpretation of the superreflection effect is related to the explanation of the sign of the transmitted wave energy. Consider initially the momentum density of a monochromatic sound wave in a resting medium.

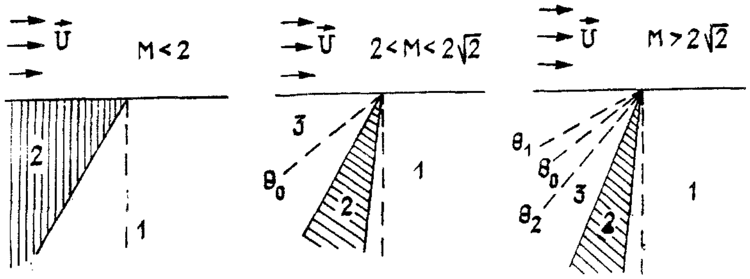

Fig. 3. Different regimes of sound reflection from a tangential discontinuity. The following regions of incidence angles are shown: 1) normal reflection; 2) total reflection; 3) superreflection.

Assuming that the mean Euler velocity is $\langle V\rangle = 0$ (no induced mean flows), we obtain the mean momentum density

$$\boldsymbol{P} = (1/2)\,\mathrm{Re}\,\tilde{\rho}\boldsymbol{v}^*, \tag{10}$$

where $\tilde{\rho}$ and $\boldsymbol{v}$ are the density oscillation amplitude and the velocity, calculated within the linear problem, and $\varepsilon_0 = (1/2)[(\rho|\boldsymbol{v}|^2/2) + (|p|^2/2\rho c^2)] = (\rho\omega^2/2c^2)|\varphi|^2$ is the mean energy density (the equality of the mean potential and kinetic sound energy is taken into account) [11]. We note that the particle momentum here, as also for surface waves, is related to the particle drift (the mean Lagrangian velocity):

$$\boldsymbol{P} = \rho\,\langle V_L\rangle = \frac{1}{2}\rho\,\mathrm{Re}[(\xi^*\,i\boldsymbol{k})\boldsymbol{v}] = \frac{\rho|v|^2}{2}\frac{\boldsymbol{k}}{\omega}. \tag{11}$$

In a moving medium, taking into account that the pressure amplitude is independent of the reference system, we obtain from (7) the following energy density

$$\varepsilon = \frac{\omega}{\omega - kU}\,\frac{|p|^2}{2\rho c^2} = \frac{\rho\omega}{2c^2}(\omega - kU)\,|\varphi|^2. \tag{12}$$

For $\omega - kU < 0$ the energy density is negative. The vertical component of the energy flow density is $S_y = v_{gr\,y}\varepsilon = (\rho\omega/2)q_2|\varphi|^2$. For $q_2 < 0$ the energy flux of the transmitted wave is directed toward the discontinuity: $S_y < 0$. Thus, amplification occurs due to energy flux from the moving medium. In this case a negative energy wave emerges from the moving medium. The energy conservation law $q_1(1 - |R|^2) = q_2|T|^2$ can be derived directly from expression (9).

<u>2.2. Resonances of Supersonic Flow</u>. Consider the various reflection regimes depending on the incidence angle θ and the Mach number $M = U/c$. Representing the wave vector of the incident wave in the form $\boldsymbol{k}_0 = (k, q) = (\omega\sin\theta/c, \omega\cos\theta/c)$, we rewrite (9) in the form

$$R = \frac{\cos\theta - [1 - \sin^2\theta(1 - M\sin\theta)^{-2}]^{1/2}}{\cos\theta + [1 - \sin^2\theta(1 - M\sin\theta)^{-2}]^{1/2}},$$
$$T = \frac{2\cos\theta(1 - M\sin\theta)^{-1}}{\cos\theta + [1 - \sin^2\theta(1 - M\sin\theta)^{-2}]^{1/2}}. \tag{13}$$

Three different reflection regimes are possible (see Fig. 3):

1) normal reflection ($q_2 > 0$, $|R| < 1$) for $\sin\theta < (M + 1)^{-1}$;

2) total reflection ($\mathrm{Re}\,q_2 = 0$, $|R| = 1$) for $(M + 1)^{-1} \le \sin\theta \le 1$ if $M \le 2$, and for $(M + 1)^{-1} \le \sin\theta \le (M - 1)^{-1}$ if $M > 2$;

3) superreflection ($q_2 < 0$, $|R| > 1$) for $M > 2$ and $\sin\theta > (M - 1)^{-1}$.

In the latter case there exists a resonance incidence angle $\theta_0 = \arcsin(2/M)$, for which $|R| = \infty$. Under this angle we have spontaneous Cherenkov radiation of a vortex sheet, moving with velocity $U/2$ [13, 14]. A wave of negative energy is emitted during the process of spontaneous emission in the moving medium, and a wave of positive energy – in the resting one. TD oscillations are neither damped nor amplified, and the sound energy emitted in the resting medium is drawn from the whole moving medium.

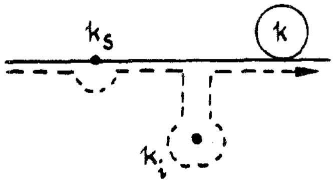

Fig. 4. The integration contour in the complex k-plane.

For $M \geq 2\sqrt{2}$ the TD becomes stable [11], and the Kelvin-Helmholtz surface modes transform into running modes over the wave discontinuity with a phase velocity (relative to the vortex sheet) of $V = \pm c[1 + (M^2/4) - \sqrt{1 + M^2}]^{1/2}$. Since $V + (U/2) > c$, Vavilov–Cherenkov radiation must be generated [1]. Perturbations traveling over the TD emit sound at angles $\theta_{1,2}$, for which

$$\sin\theta_{1,2} = \left[\frac{M}{2} \pm \left(1 + \frac{M^2}{4} - \sqrt{1+M^2}\right)^{1/2}\right]^{-1} \tag{14}$$

The presence of negative energy waves in a supersonic TD can lead to various dissipative instabilities. We also note that the presence of a boundary supplements the acoustic inverse relation to TD, amplifying the reflected sound, and, obviously, rendering the flow unstable. An instability of this type was found in supersonic boundary layers [4].

2.3. Discontinuity Excitation by the Incident Wave. Considering planar monochromatic waves, we have assumed that sound and the eigenoscillations of discontinuity are linearly independent modes. At the same time sound waves from a real source, possessing finite sizes and finite duration, excite discontinuity instability. Consider here radiation of a monochromatic point source of unit mass at distance h from the TD [18].* The wave equations in the resting and moving media are, respectively,

$$\Delta\varphi_1 + \left(\frac{\omega}{c}\right)^2 \varphi_1 = \delta(x, y+h), \quad \Delta\varphi_2 - \frac{1}{c^2}\left(-i\omega + U\frac{\partial}{\partial x}\right)^2 \varphi_2 = 0. \tag{15}$$

Taking into account the boundary conditions on the TD, one easily obtains a solution by the Fourier transform in the coordinate x:

$$\varphi = \varphi_0 + \frac{1}{2\pi}\int \frac{idk}{2q_1} e^{ikx - iq_1(y-h)} R(\omega, k), \tag{16}$$

where φ_0 is the source field in an unbounded resting medium, and $R(\omega, k)$ is defined by expression (9).

The integration contour in the complex k-plane must be selected by using the causality principle. Keeping in mind the solution of the initial problem by the Laplace method, we must consider complex ω values corresponding to growing waves, i.e., located sufficiently far in the upper ω half-plane. In this case the poles of the reflection coefficient $R(\omega, k)$ are located in the upper complex k half-plane. In this case the integration can be carried out over the real k axis. In order to continue analytically the solution obtained to real ω, it is necessary to deform the integration path in the complex k-plane, supplementing the real axis by loops surrounding the poles k_i in the lower half-plane, and by arcs surrounding the poles k_S on the real axis (Fig. 4).

The poles k_i correspond to eigenoscillations of the discontinuity, increasing along the x axis. Thus, the full solution of the point source problem includes not only traveling sound waves [obtained by integration over the real axis in (16)], but also a surface wave at the discontinuity increasing along x.

The solution thus obtained makes it possible to establish the validity limits of the results, related to reflection of monochromatic plane waves from a TD instability. Indeed, upon moving away from the point source ($h \to \infty$) an incident cylindrical wave tends to a planar wave near the given direction. In this case the effectiveness of exciting a surface wave decreases exponentially. We also note that the lines of equal amplitude of the surface wave are rays with slope $\mathrm{tg}\,\theta_c = \mathrm{Im}\,q/\mathrm{Im}\,k$. For incidence angles $\theta > \theta_c$ the solution in the

*The interesting features generated during TD excitation by a cylindrical pulse were treated in [19].

form of transmitted and reflected waves loses its meaning, since it exists on the background of an exponentially increasing solution of the surface wave type. At the same time, for $\theta < \theta_c$ the surface wave can be neglected. At low velocities ($M \to 0$) we have $\theta_c = 45°$, while for $M \to 2\sqrt{2}$ the growing waves vanish and $\theta_c \to 90°$.

The solution (16) makes it possible to calculate the acoustic impedance of source emission near the TD: $r_a = -\mathrm{Im}[\rho\omega\varphi(x = 0, y = -h)]$. For subcritical discontinuities at $M \to 0$ we have $q_{1,2} = ik$, and the reflection coefficient has a simple form: $R = [(\omega - kU)^2 - \omega^2]/[(\omega - kU)^2 + \omega^2]$. In this case the integral over the real axis in (16) obviously does not contribute to r_a. The radiation impedance is determined by the excited surface wave, corresponding to a contour integral around the poles $k_i = (1 - i)\omega/U$. As a result we have

$$r_a = -\frac{1}{4}\rho\omega e^{-2\omega h/U}\sin\left(\frac{2\omega h}{U} + \frac{\pi}{4}\right). \qquad (17)$$

The quantity r_a depends on the parameter $\omega h/U$, and can change sign. The latter fact makes it possible to explain the mechanism of self-excitation of several types of spectra [6, 20]. In fact, if as a source mass one takes, for example, a Helmholtz resonator, for $r_a < 0$ the oscillations in the resonator are amplified.

We note that for self-excitation of a resonator the important feature is not so much the flow instability, but the presence of self-oscillations of the flow, its inertial property. In particular, the source can have a negative radiation impedance, excited by neutrally stable oscillations in the flow due to the development of these oscillations at the length of flight near the source [20]. This mechanism is similar to the self-excitation mechanism of electron SHF devices, where the evolvement of perturbations in the electron flow at the length of flight h is determined by the parameter $\omega h/U$ and leads to electron clustering. In the presence of these unstable oscillation modes of the electron flow the buildup increment of the electromagnetic resonator is determined, as in hydrodynamics, by an integral in the complex wave number plane over a contour, surrounding the poles corresponding to waves increasing with the flow from below [21].

2.4. Superreflection of Internal Gravity Waves. The nature of reflection of dispersionless sound waves is determined only by the incidence angle. Below we consider waves with dispersion, whose reflection depends on their frequency. Interesting and practically important examples are internal gravity waves (IGW), propagating in a stratified medium in a gravity field.

The equations for two-dimensional oscillations of a layered, incompressible medium are [11, 22]

$$\dot{u} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \quad \dot{v} + \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{g\tilde{\rho}}{\rho} = 0,$$
$$\dot{\tilde{\rho}} + v\frac{d\rho}{dy} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (18)$$

As a rule, for IGW one can use the Boussinesq approximation, corresponding to the limit $d\rho/dy \to 0$, $g \to \infty$, $N \equiv [(-g/\rho)(d\rho/dy)]^{1/2} = \mathrm{const}$. In this case the density variation at a wavelength scale becomes unimportant, but remains an increasing force. In this approximation the derivative $d\rho/dy$ appears only in the Brent–Väisälä frequency N, while in the coefficients of the equations one can put $\rho = \mathrm{const}$. From (18) one then obtains a dispersion equation for IGW and their group velocity v_{gr}:

$$\omega^2 = \frac{N^2 k^2}{k^2 + q^2} = N^2 \sin^2\theta, \quad \mathbf{v}_{gr} = \frac{Nq}{(k^2 + q^2)^{3/2}}(q, -k) = \frac{N\cos\theta}{k_0}(\cos\theta, -\sin\theta), \quad \mathbf{k}\mathbf{v}_{gr} = 0. \qquad (19)$$

We note that only IGW with $\omega \le N$ can propagate.

Consider IGW reflection from a TD in a medium with N = const. Matching the solutions $p_1 = \exp(iq_1 y) + R\exp(-iq_1 y)$ and $p_2 = T\exp(iq_2 y)$ by means of the boundary conditions at the TD, we obtain

$$R = \frac{q_1(N^2 - \omega^2)^{-1} - q_2[N^2 - (\omega - kU)^2]^{-1}}{q_1(N^2 - \omega^2)^{-1} + q_2[N^2 - (\omega - kU)^2]^{-1}},$$

$$T=\frac{2q_1(N^2-\omega^2)^{-1}}{q_1(N^2-\omega^2)^{-1}+q_2[N^2-(\omega-kU)^2]^{-1}}, \tag{20}$$

where $q_1 = -k\sqrt{N^2/\omega^2 - 1}$, $q_2 = -k\sqrt{N^2/(\omega - kU)^2 - 1}$ (it is taken into account that for IGW $q_1 v_{gr\,y} < 0$). The energy conservation law following from (20) $q_1(N^2 - \omega^2)^{-1}(1 - |R|^2) = q_2[N^2 - (\omega - kU)^2]^{-1}|T|^2$ is also easily obtained from expression (19) for $v_{gr\,y}$ and the equation for the IGW energy density in a resting medium [22]

$$\varepsilon_0=\frac{\rho\langle u^2+v^2\rangle}{2}+\frac{N^2\langle\zeta^2\rangle}{2}=\frac{N^2k^2}{\omega^2(N^2-\omega^2)}|p|^2 \tag{21}$$

with account of Eq. (7) and the relationship $S_y = v_{gr\,y}\varepsilon$.

The radiation condition in a moving medium $v_{gr\,y}=-\dfrac{N^2k^2}{(k^2+q^2)^2}\dfrac{q}{\omega-kU}>0$ with $\omega - kU < 0$ forces to select the branch $q_2 > 0$. Taking into account that $q_1 < 0$, we obtain in this case $|R| > 1$. As follows from (7), the transmitted wave carries negative energy in this case.

The classification of the various reflection regimes is conveniently carried out in terms of the dimensionless parameter $s = kU/\omega = (k_0U/N)(\sin\theta/|\sin\theta|)$, in terms of which are expressed the wave characteristics in a moving medium and the coefficients R, T:

$$\omega-kU=N|\sin\theta|(1-s),\quad q_2=-\frac{k\sqrt{1-\sin^2\theta(1-s)^2}}{|\sin\theta|(1-s)},$$

$$v_{gr\,y}=\frac{N\sin\theta}{k_0}(1-s)^2\sqrt{1-\sin^2\theta(1-s)^2}, \tag{22}$$

$$R=\frac{(1-s)\sqrt{1-\sin^2\theta(1-s)^2}-\cos\theta}{(1-s)\sqrt{1-\sin^2\theta(1-s)^2}+\cos\theta},$$

$$T=\frac{2(1-s)\sqrt{1-\sin^2\theta(1-s)^2}}{(1-s)\sqrt{1-\sin^2\theta(1-s)^2}+\cos\theta}.$$

Superreflection is possible for $s > 1$, $\sin\theta < (s - 1)^{-1}$. Total reflection ($\mathrm{Re}\, q_2 = 0$) occurs for $s > 2$ in the region $\sin\theta > (s - 1)^{-1}$, and for $s < 0$ in the region $\sin\theta < -(1 + |s|)^{-1}$. In the remaining cases we have normal reflection.

Of particular interest are resonances ($|R| = \infty$), possible only for $s > 1$. One of them is determined by the condition $s = 2(\omega = kU/2)$, and corresponds to Cherenkov radiation of IGW vortex sheets. The irradiation angle is determined in this case by the relation $\sin\theta = \omega/N = kU/2N$. Two other resonances are determined by the condition $\mathrm{ctg}\,\theta = s - 1$, corresponding to the dispersion equation $\omega^2 + (\omega - kU)^2 = N^2$ for intrinsic waves on the TD. These modes are stable for $|k| < \sqrt{2}\, N/U$, and transform to Kelvin-Helmholtz modes for $N \to 0$.

2.5. Rossby Waves on a Tangential Discontinuity. The presence of a hydrodynamic flow with sufficiently high velocity, capable of "overtaking" waves incident on it, does not guarantee the possibility of superreflection. For illustration we consider incidence of Rossby waves on a TD at the β-plane [23].

The linearized equations of motion of an incompressible fluid at a plane rotating with angular velocity f/2 are

$$\dot{u}-fv+\frac{\partial p}{\partial x}=0,\quad \dot{v}+fu+\frac{\partial p}{\partial y}=0,\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{23}$$

where $\beta = df/dy = \mathrm{const}$ characterizes the gradient of the Coriolis force. Introducing the current function $\psi(u = \partial\psi/\partial y,\ v = -\partial\psi/\partial x)$, one easily obtains a dispersion equation and the group velocity for Rossby waves (retaining the original notations):

$$\omega=-\frac{\beta k}{k^2+q^2}=-\frac{\beta}{k_0}\sin\theta,\quad v_{gr}=\frac{\beta}{(k^2+q^2)^2}(k^2-q^2,2kq)=\frac{\beta}{k_0^2}(-\cos 2\theta,\sin 2\theta). \tag{24}$$

The boundary conditions at the TD-continuity are for displacement $\zeta = k\psi/(\omega - kU)$ and for pressure $p + \zeta dP_0/dy = iq(\omega - kU)\psi/k$. The latter expression is easily obtained by accounting for the geostrophy condition $dP_0/dy = -fU$ for the pressure P_0 in an unperturbed

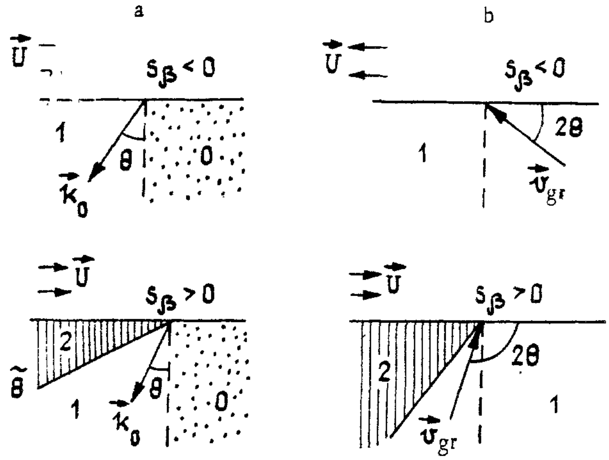

Fig. 5. Regimes of Rossby wave reflection from a tangential discontinuity. Regions of incidence angles for the wave vector (a) and the group velocity (b): 0) propagation impossible, 1) normal reflection, 2) total reflection.

flow, and by expressing the pressure amplitude in the form $p = -f\psi + (\omega/k - U)\partial\psi/\partial y$. Matching then the solutions $\psi_1 = \exp(iq_1 y) + R\exp(-iq_1 y)$ and $\psi_2 = T\exp(iq_2 y)$ (where $q_1 = \sqrt{-k^2 - \beta k/\omega}$, $q_2 = \sqrt{-k^2 - \beta k/(\omega - kU)}$), we obtain

$$R = \frac{q_1\omega^2 - q_2(\omega - kU)^2}{q_1\omega^2 + q_2(\omega - kU)^2}, \quad T = \frac{2q_1\omega(\omega - kU)}{q_1\omega^2 + q_2(\omega - kU)^2}. \tag{25}$$

The radiation condition in a moving medium with $y \to \infty$ $v_{gr\,y} = 2\beta k q_2 (k^2 + q_2{}^2)^{-1} > 0$ determines the relation $q_2/q_1 > 0$. Thus, superreflection of Rossby waves from TD is impossible. This is explained by the fact that the energy density of the transmitted wave in a moving medium is

$$\varepsilon = \frac{\omega}{\omega - kU}\varepsilon_0 = \frac{\omega}{\omega - kU}(k^2 + q_2^2)|\psi|^2 = \frac{(k^2 + q_2^2)^2}{(k^2 + q_1^2)}|\psi|^2 > 0. \tag{26}$$

In Fig. 5 we show the various reflection regimes for $\beta > 0$, depending on the parameter $s_\beta = [1 + (k^2 + q_1{}^2)U\beta^{-1}]^{-1}$. The regions of normal and total reflection are separated by the angle $\theta = \arcsin\sqrt{s_\beta}$.

The condition $|R| = \infty$ provides the dispersion equation for the intrinsic waves at the TD in a rotating medium:

$$(\omega - kU)^2\sqrt{k^2 + \beta k/(\omega - kU)} + \omega^2\sqrt{k^2 + \beta k/\omega} = 0. \tag{27}$$

For the dimensionless phase velocity $c = 2\omega/kU - 1$ one easily obtains from Eq. (27) the cubic equation $sc^3 + 3c^2 + sc - 1 = 0$ (where $s = 2k^2/U/\beta$), whose discriminant is $\Delta = -4s^4 + 36s^2 - 108 < 0$, and, consequently, it has two complex-conjugate roots. Thus, TD instability is also retained in the presence of rotation, and for $s \to 0$ it transforms to a Kelvin–Helmholtz instability.

3. VORTEX OSCILLATIONS

3.1. Algebraic Method for Cylindrical Vortices. The wave properties considered above in shear flows have close analogy in flows with closed current lines, cylindrical vortices. To study small perturbations in axially symmetric flows an algebraic method is developed, based, as in plane-parallel flows, on approximating the velocity distribution by a profile with piecewise constant vorticity and matching of analytic solutions at region boundaries [24].

In polar coordinates (r, φ) the velocity of stationary flow in a cylindrical vortex equals $V = (0, r\Omega)$, while the pressure is $P_0 = \int \rho r\Omega^2 dr$. The linearized Navier–Stokes equations for the perturbation amplitudes $\exp(-i\omega t + in\varphi)$ of velocity $v = (v, u)$ and pressure p in a homogeneous fluid (ρ = const) are

$$-i(\omega-n\Omega)v - 2\Omega u + \frac{1}{\rho}\frac{dp}{dr} = \nu\left(\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{n^2+1}{r^2}v - \frac{2inu}{r^2}\right),$$

$$-i(\omega-n\Omega)u + 2\Omega v + rv\frac{d\Omega}{dr} + \frac{inp}{\rho r} = \nu\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{n^2+1}{r^2}u + \frac{2inv}{r^2}\right), \quad -i(\omega-n\Omega)\frac{p}{\rho c^2} + \frac{dv}{dr} + \frac{v}{r} + \frac{inu}{r} = 0. \tag{28}$$

In an incompressible fluid one can introduce the curent function ψ, in whose terms are expressed the velocity components and the pressure: $u = d\psi/dr$, $v = -in\psi/r$, $p = \rho[r(\omega/n - \Omega)\cdot d\psi/dr + \alpha\psi]$, where $\alpha \equiv 2\Omega + rd\Omega/dr$ is the vorticity of the unperturbed flow. For the current function we have the analog of the Orr–Sommerfeld equation [4]

$$\hat{L}\psi + \frac{n d\alpha/dr}{r(\omega-n\Omega)}\psi = \frac{i\nu n}{\omega-n\Omega}\hat{L}^2\psi, \tag{29}$$

where $\hat{L} \equiv d^2/dr^2 + r^{-1}d/dr - n^2/r^2$. For $\nu = 0$ Eq. (29) is similar to the Rayleigh equation, and contains the singular points $\omega - n\Omega = 0$ with a coefficient proportional to the derivative of the vorticity $d\alpha/dr = rd^2\Omega/dr^2 + 3d\Omega/dr$. In the Rayleigh equation the residue of the coefficient at the singular point had the same meaning, and was proportional to the second derivative of the velocity profile.

For ideal fluid flows with constant vorticity, in which $\Omega = \Omega_0 + \kappa/r^2$, one can find a solution of (29) in the form $\psi = \psi_a \equiv Ar^n + Br^{-n}$. The algebraic method uses angular velocity profiles, consisting of several parts with homogeneous vorticity, bounded by tangential discontinuities or velocity nodes. For this "piecewise" profile it is easy to find a solution matching expressions of the type ψ_a in each of the parts by means of boundary conditions at the discontinuities $r = a$:

$$\left[p + \zeta\frac{\partial P_0}{\partial r}\right]_{r=a-0}^{r=a+0} = 0, \quad [\zeta]_{r=a-0}^{r=a+0} \equiv \left[\frac{v}{-i(\omega-n\Omega)}\right]_{r=a-0}^{r=a+0} = 0. \tag{30}$$

Staying within the algebraic method, one can also estimate by a model the effect of viscosity on vortex oscillations. For this it is sufficient to consider a discontinuity between a viscous and an ideal fluid, i.e., take into account the viscosity, for example, only in the vortex core.

Account of viscosity with $d\alpha/dr = 0$ does not change the solutions ψ_a, since the latter satisfy Eq. (29) identically. Enhancement of the order of the equation with $\nu \neq 0$ leads to the appearance of additional linearly independent solutions, which for large Reynolds numbers Re oscillate quickly and decay. Restricting ourselves to leading order terms only in Re, we obtain from (28), (29)

$$-i(\omega-n\Omega)\psi = \nu\frac{d^2\psi}{dr^2}, \quad p = \rho\alpha\psi. \tag{31}$$

The solution, quickly decaying for $r < a$, is of the form $\psi = G\exp[q_\nu(r-a)]$, where $q_\nu = \sqrt{i(\omega-n\Omega)/\nu}$, $\mathrm{Re}\, q_\nu > 0$. The boundary conditions at the TD are the continuity of the displacement $\zeta = iv/(\omega - n\Omega)$ and continuity of the normal components of the flow momentum to the moving boundary, which can be expressed in terms of the values of the stress tensor σ_{ik} and its derivatives at the boundary $r = a$,

$$\sigma_{rr} + \zeta\frac{\partial\sigma_{rr}}{\partial r} - \frac{\sigma_{r\varphi}^{(0)}}{a}\frac{\partial\zeta}{\partial\varphi} \equiv \left[-p + 2\rho\nu\frac{dv}{dr} - \rho a\Omega^2\zeta - in\rho\nu\frac{d\Omega}{dr}\zeta\right]e^{q_\nu(r-a)},$$

$$\sigma_{r\varphi} + \zeta\frac{\partial\sigma_{r\varphi}}{\partial r} - \frac{\sigma_{\varphi\varphi}^{(0)}}{a}\frac{\partial\zeta}{\partial\varphi} \equiv \rho\nu\left[\frac{du}{dr} - \frac{u}{r} + \frac{inv}{r} + \left(r\frac{d^2\Omega}{dr^2} + \frac{d\Omega}{dr}\right)\zeta\right]e^{q_\nu(r-a)}, \tag{32}$$

where $\sigma_{ik}^{(0)}$ is the unperturbed stress tensor. These conditions make it possible to match nonviscous solutions in the external vortex region and a linear combination of nonviscous and damped viscous solutions in the vortex core.

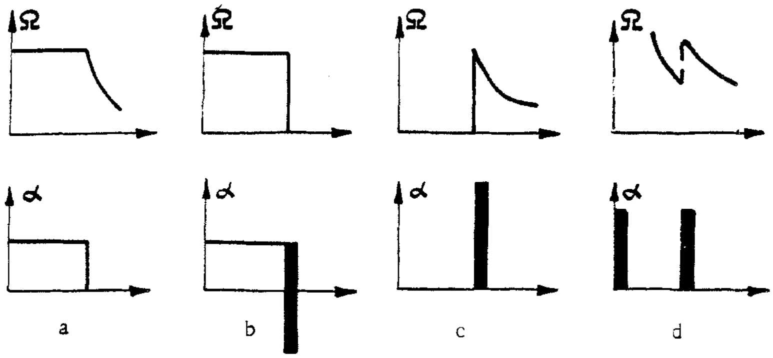

Fig. 6. Angular velocity profiles and vorticities for various cylindrical vortices.

3.2. Vortex Stability. For vortices in an incompressible fluid one can prove a statement similar to the Rayleigh theorem [25]: Perturbations increasing with time can exist only in the presence of vorticity extremum points in the angular velocity profile, where $d\alpha/dr = 0$ (in plane-parallel flow this is an inflection point of the velocity profile). Examples of such vortices with nonmonotonically varying vorticity are illustrated in Fig. 6 (profiles b, c, d). In particular, for a cylindrical TD (Fig. 6c) one can find an eigenfunction in the form

$$\psi = \begin{cases} A(r/a)^{|n|}, & r \leq a \\ B(r/a)^{-|n|}, & r > a \end{cases}. \tag{33}$$

Matching solutions at the TD by means of the boundary conditions (30), we obtain the dispersion equation $\omega^2 + (\omega - n\Omega_0)^2 = |n|\Omega_0{}^2$, which transforms to the Kelvin–Helmholz equation for $n \to \infty$ (in this case $n\Omega_0 \to kU$). Its solution $\omega = \frac{1}{2}\Omega_0(n \pm \sqrt{2|n| - n^2})$ corresponds to unstable modes for all $|n| \geq 2$. The mode $n = 1$, corresponding to a vortex shift, is, as whole, obviously, neutrally stable.

For Kelvin vortices (Fig. 6a) one easily derives the dispersion equation $(\omega - n\Omega_0)[\omega - (n - 1)\Omega_0] = 0$, describing neutrally stable oscillations. This equation can be generalized, taking into account viscosity in the vortex core. Using the algebraic method described above, one can obtain in first approximation in the parameter $Re = \nu/\Omega_0 a^2$ [24]

$$\omega = (n-1)\Omega_0 - \frac{2i\nu}{a^2} n(n-1), \tag{34}$$

which corresponds to damped oscillations.

We interpret this result, taking into account, however, that in a medium rotating faster than the angular phase velocity of azimuthal waves ($\omega - n\Omega_0 < 0$) the dissipation is negative. The vortex oscillation energy

$$E = \frac{1}{2}\rho \,\mathrm{Re} \int_0^\infty 2\pi r \left(\frac{|u|^2 + |v|^2}{2} + r\Omega u\right) dr =$$
$$= \pi\rho \int_0^a \frac{|u|^2 + |v|^2}{2} r\,dr + \pi\rho \int_a^\infty \frac{|u|^2 + |v|^2}{2} r\,dr + \pi\rho \,\mathrm{Re} \int_a^{a+\zeta} r^2\Omega_0 u^* dr - \pi\rho \,\mathrm{Re} \int_a^{a+\zeta} \Omega_0 a^2 u^* dr \tag{35}$$

must also be negative. Indeed, substituting solutions for amplitudes of type (33), one easily obtains (see [26])

$$E = -\pi a^4 \Omega_0^2 \frac{n-1}{n} \rho |\zeta_0/a|^2, \tag{36}$$

where ζ_0 is the displacement amplitude of the vortex core boundary.

The negative energy of the vortex eigenoscillations renders its radiational instability in a compressible medium possible [27]. For $M \equiv \Omega a/c \ll 1$ the oscillations are near those in the incompressible case for $r \leq a$, but the corresponding nonstationary motions emit sound into the wave zone $r \geq \lambda$. As a result of energy selection by the emerging sound waves, the oscillation amplitude increases.

3.3. Sound Amplification by Vortices. The presence of negative energy waves in a rotating medium makes it possible, in principle, to amplify sound waves incident on a vortex. To realize this possibility, however, a sink of negative energy is needed, whose role in a plane-parallel TD is played by departure of transmitted waves at infinity. For vortices, for which the core region is, obviously, finite, this possibility does not exist. Here, however, is possible amplification of scattered sound due to the change of sign of viscous dissipation in the vortex core.*

Consider scattering of sound by a vortex of size $a \ll \lambda = 2\pi c/\omega$ [24, 28]. The field in the far zone

$$p = \exp\left(\frac{i\omega r\cos\theta}{c}\right) + f(\theta)\, r^{-1/2} \exp\left(\frac{i\omega r}{c}\right) = \left(\frac{2\pi\omega r}{c}\right)^{-1/2} \times$$
$$\times \sum_{n=-\infty}^{\infty} i^n e^{in\varphi} \left\{ \exp\left[-i\left(\frac{\omega r}{c} - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right] + R_n \exp\left[i\left(\frac{\omega r}{c} - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right]\right\} \tag{37}$$

is determined by the reflection coefficients of cylindrical harmonics $r_n = 1 + (2\pi i\omega/c)^{1/2} f_n$, where $f(\theta) = \sum_n f_n \exp(in\theta)$ is the scattering amplitude. The energy exchange between the wave and the vortex is determined by the quantities $|R_n|$, which can be found in first approximation in the parameter $\mu \equiv a/\lambda \ll 1$. For this it is sufficient to match the solution at $r \ll \lambda$, obtained by the algebraic method, and the solution at $r \gg a$ in the form of a sum of incident and reflected cylindrical waves. For a Kelvin vortex with a viscous core one easily obtains by this method [24, 28]

$$1 - |R_n|^2 = \frac{8\pi\nu/a^2}{(|n|-1)!\,(|n|-2)!} \left(\frac{\omega a}{2c}\right)^{2|n|} \frac{\omega - n\Omega}{[\omega - (n-1)\Omega]^2}. \tag{38}$$

For $\omega - n\Omega < 0$ sound is amplified: $|R_n| > 1$. Sound amplification by a rotating viscous vortex is the acoustic analog of the effect considered in [29], where the possibility was shown of amplifying electromagnetic waves during scattering by a rotating conducting cylinder, as well as of gravity waves by a collapsing rotating body.

It must be noted, however, that the mechanism of viscous sound dissipation in vortex flow does not reduce to simple absorption, but is determined by a linear transformation in quickly damped vortex waves.

In conclusion we note that, being restricted in the present study to TD type modes, we have excluded from consideration a wide range of problems related to the resonance mechanism of amplified waves, interacting with particles in critical layers (see, for example, [4, 6]). It is also interesting to initiate studies in instability mechanisms [8, 30] and nonlinear TD dynamics [31], where negative energy waves play an essential role.

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EFFECTIVE SATURATION OF ABSORPTION IN A MAGNETOSPHERIC PLASMA MASER

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The properties of the quasilinear interaction of whistler and Alfven waves with planetary radiation belts are discussed. It is shown that quasilinear relaxation can lead to an increase in the increment of cyclotron instability at the leading front of an electromagnetic pulse. This corresponds to the effective saturation of absorption and makes it energetically advantageous for the noise emission to be divided into separate electromagnetic pulses. Peculiarities of the manifestation of fast and slow, compared with the pulse length, effective saturation of absorption are discussed.

Research in recent years has shown that the regions of the radiation belts of the earth and Jupiter, if cyclotron instability develops in them, are largely similar in their physical properties to laboratory masers and lasers. In a magnetospheric plasma maser (MPM) the relatively dense magnetized plasma and the conjugate ends of a magnetic trap form a quasioptical resonator for electromagnetic waves. This circumstance clarifies the sense in which the term maser is used in the present case. Here a microwave nature is not understood literally but as smallness of the wavelength compared with the scale of the resonator. In turn, the active

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