Propagation of waves in hydrodynamic shear flows

Yu. A. Stepanyants and A. L. Fabrikant

Institute of Applied Physics of the Academy of Sciences of the USSR, Gor'kiĭ
Usp. Fiz. Nauk **159**, 83–123 (September 1989)

A range of phenomena connected with the propagation of waves in hydrodynamic shear flows is studied. The problem of calculating the energy and momentum of a wave packet in a moving medium is discussed in detail. It is shown that in many cases the momentum of a body moving in a liquid can be correctly calculated only if the compressibility of the medium is taken into account. The phenomenon of super-reflection of waves from the interface between moving media—the fact that the amplitude of the reflected wave can be much greater than the amplitude of the incident wave—is described. An interpretation of this phenomenon is given based on the concept of waves with negative energy. It is also shown that the reflected wave can be amplified when the sign of the dissipation in the moving medium changes. The behavior of different types of waves on a tangential discontinuity of the velocity is described (surface and internal waves as well as Rossby waves are studied). A separate section is devoted to resonant interaction between waves and the flow. Here the plasma-hydrodynamic analogy and its generalization to the case of stratified media are discussed. Resonance amplification in shear flows is studied for sound waves, surface waves on water, and internal gravity waves. The interaction of waves with vortices is discussed briefly. An algebraic method for solving problems is described for cylindrical vortices. Different mechanisms of amplification of sound by vortices are examined.

1. INTRODUCTION

Oscillations and waves in hydrodynamic flows can be studied from different perspectives. First, there is the problem of wave generation by shear flows; this is one of the basic problems in the theory of hydrodynamic instability. The simplest types of instabilities for flows of the type of tangential discontinuity of the velocity were discovered during the last century (see, for example, Ref. 1); a qualitative explanation of these instabilities is given in Refs. 2 and 3. The study of wave disturbances in a flow with a continuous velocity profile, which was initiated in the fundamental works of Rayleigh and Heisenberg,[4,5] turned out to be a very difficult problem. The asymptotic theory of equations with singular perturbations[6,7] as well as numerical methods[1,8] enabling detailed analysis of a wide class of flows were developed to solve this problem.

The efforts made to overcome the mathematical difficulties arising in the theory, however, often did not clarify the physical picture of the processes occurring. In this respect the analogy that has now been developed between wave-flow interaction in hydrodynamics and the corresponding phenomena in electrodynamics, plasma physics, and electronics has turned out to be very useful for developing the intuition, for gaining a qualitative understanding of the phenomena, and often also for choosing the optimal computational methods. This analogy, first pointed out by Case,[9] was later developed in different directions.[10–20] As a result it was established that concepts such as waves with negative energy, negative dissipation, resonance between a wave and a flow, etc., which form the basis for the theory of plasma instabilities, also work successfully in hydrodynamic problems. A physical explanation for the well-known results of hydrodynamics (Lin's rules for constructing integration contours round singular resonance points in shear flows, the instability of flows with a point of inflection in the velocity profile,[14] Miles' theory of the generation of wind waves[20]) has been given based on the plasma-hydrodynamic analogy, and a number of new hydrodynamic effects have also been studied—analogs of linear and nonlinear Landau damping in shear flows,[20] quasilinear interaction and induced scattering of waves by particles for wind waves in the ocean,[21,22] cyclotron absorption accompanying scattering of sound by vortices,[23,24] and others. The analogy also "works" in the other direction: it makes it possible to study, for example, some types of plasma instabilities, starting from known results of hydrodynamics.[14] Studies of this type single out some effects which are common to systems of different physical nature and permit constructing a unified, general-physical language that facilitates the exchange of ideas and methods from different areas of physics.

This approach also turns out to be useful for studying a wide class of different problems—refraction, absorption, and amplification of externally generated waves in hydrodynamic flows. At the present time the mechanisms of the interaction of waves with shear flows have been studied in the physics of the atmosphere and the ocean for wind waves,[25] internal gravity waves,[26] and other waves of the hydrodynamic type.[27] Analogous processes for sound waves are under study in the rapidly developing field of aerohydroacoustics (in connection with the problem of the generation and absorption of aerodynamic noise and other problems of practical importance[28]). The study of the propagation of electromagnetic waves in a medium with shear flows (see, for example, Refs. 29 and 30) is of great interest for different problems in plasma physics (both in the laboratory and in space), microwave electronics, and magnetohydrodynamics.

Two basic mechanisms for amplification (absorption) of waves propagating in flows of an ideal liquid can be distinguished. One of them is determined by the interaction with negative-energy waves in a moving medium[31] (this corresponds to the so-called "hydrodynamic" instability of plas-

 0038-5670/89/090783-23$01.80

ma flow), while the other is determined by resonance interaction with nonequilibrium particles (the analog of the mechanism of kinetic instability of plasma waves). A dissipative mechanism, when the amplification of waves in a flow appears only in the presence of viscous dissipation, radiation losses, etc., is also possible— this corresponds to the mechanism of dissipative instability of waves in a plasma.

The unified approach, which has emerged, to the problems of the propagation of waves of different physical nature in shear flows makes it possible to summarize investigations of different oscillatory and wave phenomena in hydrodynamic flows and also to propose possible directions for further investigations.

2. THE ENERGY AND MOMENTUM OF WAVES IN A MOVING MEDIUM

The laws of conservation of the energy and momentum of waves are widely employed in the interpretation of results in the theory of waves in flows. The definition of the energy and momentum of waves in a medium, however, usually circumvents some aspects of these concepts which are not completely understood. An unsatisfactory situation arises in connection with the well-known fact that in calculating quantities that are second-order infinitesimals in the wave amplitude, which the momentum and energy, generally speaking, are, the changes in the average parameters of the medium and, in particular, the changes induced by the wave flow must obviously be taken into account. In so doing an ambiguity arises in separating the fields of physical variables into the wave field and the motion of the medium. In electrodynamics these questions arise in the detailed discussion of momentum by Abraham and Minkowski for electromagnetic waves in a dielectric.[32] In hydrodynamic problems the question of the momentum of quasimonochromatic waves is also quite muddled (see, for example, Ref. 33).

This section is devoted to a detailed explanation of the concepts of the momentum and energy of waves in a medium and their physical meaning is clarified. Analysis of these questions, based on comparison with classical hydrodynamic flows[34], leads to a definition of the momentum of a quasimonochromatic wave which fits in a natural manner into the general physical ideas of quasiparticles as quanta of wave excitations in a medium, and takes into account at the same time the existence of wave-induced average flows, which are responsible for a number of physical effects.

Thus the approach presented below gives a physical basis for the traditional (as a rule, formal) use of the concepts of quasienergy and quasimomentum.

2.1. Wave momentum and quasiparticles

The energy and momentum of waves should, strictly speaking, be defined within the framework of a nonlinear problem and should take into account the changes brought about in the average values of the physical fields[35] by the "detection" of the wave, i.e., as a result of the nonlinear generation of a low-frequency perturbation whose form is identical to that of the envelope of the high-frequency field. Usually, however, a different approach is employed. After the linearized problem is solved the quasienergy density $\mathscr{E}$ and the quasimomentum $\mathbf{P}_W$ are defined in terms of the adiabatic invariant (wave action) N, which in the quantum approach is the number of quasiparticles:

$$\mathscr{E} = \omega N, \quad \mathbf{P} = \mathbf{k}N. \tag{2.1}$$

The quantity N does not depend on the choice of coordinate system, and it is therefore convenient to calculate the quasienergy $\mathscr{E}_0 > 0$ in the medium at rest, after which the quasienergy in the medium moving with velocity $\mathbf{U}$ can be determined using a Galilean transformation:

$$\mathscr{E} = \mathscr{E}_0 + \mathbf{P}_W\mathbf{U} = \mathscr{E}_0 \frac{\omega}{\omega - \mathbf{kU}}. \tag{2.2}$$

We note that for $\omega - \mathbf{kU} < 0$ the quantity $\mathscr{E}$ becomes negative: it is in this sense that one talks about negative-energy waves.[35]

For a wide class of problems (in particular, linear problems) we can employ the idea of quasiparticles, relying on the formulas (2.1) and (2.2). The laws of conservation of quasienergy and quasimomentum follow from the linearized equations of motion, averaged over the phase of the wave, and are related to the uniform and steady-state nature of the undisturbed medium.[33] However in studying questions such as the interaction of waves with flows, the nonlinear effects due to the self-action of waves, etc., it is necessary to recall the basic definitions of energy and momentum as conserved quantities which are connected with the fact that the laws of motion are independent of time and location. The complete expressions for the energy and momentum following from the starting (nonlinear) system of equations must be averaged over the phase of the wave, and the part that is quadratic in the amplitude and is determined by the wave must then be separated. The quadratic terms which were dropped when the starting system was linearized can, generally speaking, make a contribution that is comparable to $\mathscr{E}$ and $\mathbf{P}_W$. The motions corresponding to these terms are induced wave flows.

If the undisturbed medium is at rest, then the induced wave flow, whose velocity is quadratic in the amplitude, obviously does not contribute to the kinetic energy. As regards the momentum of the wave motion it is convenient to separate it into a quasimomentum ("pseudomomentum" in the terminology of Ref. 33), determined on the basis of the linear approximation, and the momentum of the induced wave flow. The characteristic features of these two components can be seen in the example of gravity waves on the surface of a heavy liquid.

Two-dimensional gravity waves on deep water are described by the nonlinear boundary-value problem

$$\begin{gathered}\Delta\varphi = 0,\\ \frac{\partial\eta}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\varphi}{\partial y},\\ \frac{\partial\varphi}{\partial t} + g\eta + \frac{1}{2}(\nabla\varphi)^2 = 0\\ \text{with } y=\eta,\end{gathered} \tag{2.3}$$

where φ is the velocity potential, $\mathbf{V} = (u,v) = \nabla\varphi$, and η is the vertical component of the displacement of the fluid particles $\zeta = (\xi,\eta)$. The solutions of the linearized problem have the form

$$\varphi = \varphi_0 \exp(-i\omega t + ikx + |k|y)$$

where

$$\omega^2 = g|k|.$$

If there is no average flow ($\langle \mathbf{V} \rangle = 0$), the average Lagrangian velocity of a particle (Lagrangian drift)[25] is given by

$$\langle u_L \rangle = \langle u \rangle + \langle (\xi \nabla) V \rangle = \left\langle \xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} \right\rangle = \frac{k}{\omega} \langle u^2 + v^2 \rangle. \tag{2.4}$$

The average Lagrangian velocity decreases with depth as $\sim \exp(2|k|y)$ and determines the horizontal transport of mass:

$$S = \rho \int_{-\infty}^{0} \langle u_L \rangle \, dy = \frac{\rho}{2|k|} \langle u_L(0) \rangle = \rho \langle u(0) \eta \rangle. \tag{2.5}$$

The same mass flux can also be obtained in an Eulerian description, if the mass transfer in the layer between the crests and troughs of the waves (Stokes transport) is taken into account:

$$S = \left\langle \rho \int_{-\infty}^{\eta} u \, dy \right\rangle = \rho \langle u(0) \eta \rangle. \tag{2.6}$$

Stokes transport is uniquely related with the quasimomentum of the wave:

$$P_W = \int_{-\infty}^{0} \langle \rho u_L \rangle \, dy \equiv S = \frac{k}{\omega} \mathscr{E}. \tag{2.7}$$

The average flow induced by a quasimonochromatic wave in an ideal, incompressible liquid can be found to second order in the amplitude by averaging the boundary-value problem (2.3):

$$\begin{gathered} \Delta \Phi = 0, \\ \frac{\partial h}{\partial t} - \frac{\partial \Phi}{\partial y} = - \frac{\partial}{\partial x} \left\langle \eta \frac{\partial \varphi}{\partial x} \right\rangle_{y=0}, \\ \frac{\partial \Phi}{\partial t} + gh = 0, \end{gathered} \tag{2.8}$$

where $\Phi = \langle \varphi \rangle$ and $h = \langle \eta \rangle$ are the potential and deflection of the surface averaged over a period. Since the group velocity of the waves is small compared with the phase velocity of long-wavelength disturbances, whose scale is that of the average flow, the time derivatives can be neglected. This gives the quasistatic problem of flow under a moving distribution of mass sources on the surface $y = 0$. Obviously, the source of mass is the gradient of the mass flux S, associated with the Stokes transport. For a packet of surface waves there arises a pattern of stream lines, which in this approximation can be easily found by using the analogy with electrostatics: It is identical to the pattern formed by the lines of force of electric charges in the plane $y = 0$ (Fig. 1).

It is not difficult to show that in each section $x = \text{const}$ the Stokes transport of mass is completely compensated by the induced counterflow. Indeed, from (2.8) we have in the quasistatic approximation:

$$\frac{\partial}{\partial x} \left(\rho \int_{-\infty}^{0} \frac{\partial \Phi}{\partial x} dy + \rho \left\langle \eta \frac{\partial \varphi}{\partial x} \right\rangle_{y=0} \right) = 0. \tag{2.9}$$

The first term in parentheses gives the mass flux in the induced flow through the section $x = \text{const}$, and the second term gives the mass flux due to Stokes transport.

If the total momentum of the medium associated with the traveling wave packet is defined as the volume integral

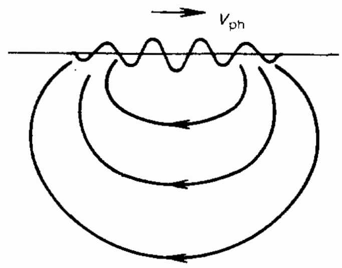

FIG. 1. The average flow induced by a packet of gravity surface waves in an ideal liquid.

$$\mathbf{I} = \int \rho \mathbf{V} \, d^3 r, \tag{2.10}$$

then for the solution of the type shown in Fig. 1 we arrive at the conclusion that the total momentum of a packet of gravity waves equals zero. In reality, the total momentum of a wave packet must be defined more carefully; this can be done by comparing with the definition of the momentum of localized flows in hydrodynamics.

2.2. The momentum of a wave packet

Localized flows in an ideal incompressible liquid are generated by the distribution of mass sources or vorticity in a bounded region. In particular, potential flow around solid bodies and localized vortices belong to this class of flows. It is not difficult to determine the transport of mass in such flows. For vortices the total mass flux through any fixed surface obviously equals zero (this is a consequence of incompressibility). For a uniformly moving sphere the mass flux through the surface S (Fig. 2) is directed backward—in the direction opposite to the velocity of the sphere $\mathbf{U}$.

At the same time the total momentum of the flow is difficult to calculate. In particular, the integral

$$I = \int \rho \mathbf{V} \, d^3 r, \tag{2.11}$$

taken over an infinite volume, as is well known, does not converge absolutely,[36] since the velocity potential at large distances has the form

$$\varphi = \left(\mathbf{A} \nabla \frac{1}{r} \right) = - \frac{(\mathbf{A}, \mathbf{r})}{r^3}, \tag{2.12}$$

while the velocity field drops off as $\sim r^{-3}$, thereby creating a nonintegrable singularity as $r \to \infty$.

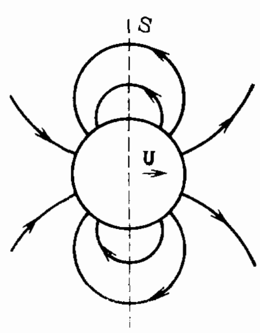

FIG. 2. Potential flow around a sphere.

The momentum of a flow is determined by studying the nonstationary processes associated with the change in the kinetic energy $dE = \mathbf{U}d\mathbf{P}$. In particular, for a sphere with radius R we have[36]

$$\mathbf{A} = \frac{\mathbf{U}R^3}{2}, \quad \mathbf{P} = \frac{4}{3}\pi\rho\mathbf{A}. \tag{2.13}$$

We note that the integral (2.11) over a spherical layer of finite thickness surrounding the sphere vanishes. The momentum of the flow (2.13) is thus fixed by the contribution of a region at *infinity*; this contribution is *uniquely* determined by the dipole moment $\mathbf{A}$.

The physical meaning of this result can be understood by replacing the potential flow around the sphere by some artificial flow generated by a distribution of sources and sinks over a sphere with radius R (Fig. 3). In the region outside the sphere the flow has the form (2.12), while inside the sphere the velocity field is uniform $\mathbf{U}_i = -\mathbf{A}/R^3$. Such a field is a solution of the well-known electrostatic problem of the polarization of a dielectric sphere in a uniform electric field.[37] Integrating over the interior region of the sphere we obtain

$$\mathbf{P}_i = \int \rho\mathbf{V}\, d^3r = \frac{4}{3}\pi\rho R^3\mathbf{U}_i = -\frac{4}{3}\pi\rho\mathbf{A}.$$

At the same time the contribution $\mathbf{P}_e$ of the external region to the total momentum is the same as for the flow around the sphere, and is determined by the expression (2.13). As a result the total momentum of the flow is

$$\mathbf{P} = \mathbf{P}_e + \mathbf{P}_i = 0.$$

We shall now study a doublet consisting of a point source and a point sink of strength Q, separated by a distance l and forming a dipole with the moment $\mathbf{A} = Q\,\mathbf{l}/4\pi$. If the source and sink appeared simultaneously at the instant $t_0 = 0$, the flow is a superposition of the field of a point source bounded by a spherical pressure jump and the field of a point sink bounded by a rarefaction jump (Fig. 4). For $t \gg l/c$ (c is the velocity of sound) the flow in a sphere with radius $r = ct$ is identical to the flow of an ideal incompressible liquid and its dipole asymptotic behavior is given by (2.12). The layers v_+ and v_-, formed by two eccentric spheres, contain the velocity field $\mathbf{V}_e = Q\mathbf{r}/4\pi r^3$. It is precisely these layers that determine the momentum of the dipole flow:

$$\mathbf{P}_e = 2\int_{v_+} \rho\mathbf{V}_e d^3r = \frac{4\pi}{3}\rho\mathbf{A}. \tag{2.14}$$

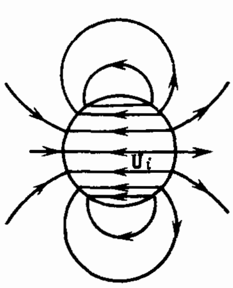

FIG. 3. Potential flow generated by a spherical distribution of sources.

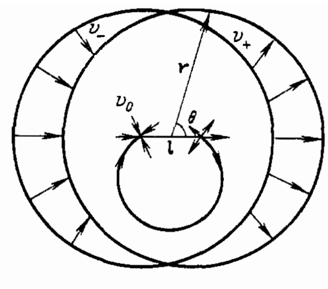

FIG. 4. Non-steady-state flow due to a dipole arising at the initial time $t_0 = 0$.

An arbitrary distribution of sources $Q(\mathbf{r})$ creates a more complicated flow, whose potential is determined by Poisson's equation $\Delta\varphi = Q(\mathbf{r})$. But if $\int Q\, dv = 0$, then in the far zone—at a distance greater than the size of the region of sources—the potential (2.12) is determined by the total dipole moment $\mathbf{A}$. The expression (2.13) gives the contribution of this far zone to the momentum $\mathbf{P}_e$. For potential flow around a sphere this expression determines the total momentum: $\mathbf{P} = \mathbf{P}_e$. For other flows the contribution $\mathbf{P}_i$ of the region of sources must be taken into account. Thus, in particular, it is possible to find the momentum of a localized vortex, for which[3]

$$\begin{gathered}\mathbf{A} = \frac{1}{8\pi}\int [\mathbf{r}\boldsymbol{\Omega}]\, d^3r, \quad \mathbf{P}_i = \frac{1}{3}\rho\int [\mathbf{r}\boldsymbol{\Omega}]\, d^3r,\\ \mathbf{P} = \mathbf{P}_i + \mathbf{P}_e = \frac{1}{2}\rho\int [\mathbf{r}\boldsymbol{\Omega}]\, d^3r,\\ \mathbf{P}_e = \frac{4}{3}\pi\rho\mathbf{A} = \frac{1}{6}\rho\int [\mathbf{r}\boldsymbol{\Omega}]\, d^3r.\end{gathered} \tag{2.15}$$

It follows from everything said above that the momentum of a localized flow in an incompressible liquid must be defined taking into account the reaction of a region at infinity to the generation of this flow. To calculate this reaction the finite velocity of propagation of disturbances must be taken into account in an explicit form. In particular, when the compressibility is taken into account part of the momentum is carried away by sound waves generated by the localized flow that arises. This result, however, does not depend on the type of propagating disturbances and on the evolution of the flow in time: the total momentum of the far zone is determined by the expression (2.13) and depends on the dipole moment $\mathbf{A}$.

The approach developed above for determining the momentum can also be extended in a natural manner to localized flows induced by quasimonochromatic wave trains. We shall examine here the simplest model of surface gravity waves, when the length of the wave packet is $l \gg H \gg \lambda = 2\pi/k$, so that the shallow-water approximation is valid for the average flows, while the propagation of the gravity waves can be studied in the deep-water approximation. The generation of a wave packet is accompanied by the appearance of a source on the leading slope of the wave and a sink on the trailing slope of the wave. The distribution of sources is related with the Stokes transport of mass, and their strength is given by [see (2.7)]

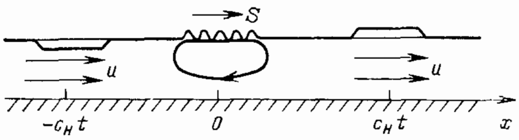

FIG. 5. The average flow arising on shallow water accompanying he generation of a packet of surface waves.

$$Q = -\frac{\partial S}{\partial x} = -\frac{\partial P_W}{\partial x}.$$

When a source and a sink appear radiation of differentials of the surface level occurs:

$$\Delta h = \frac{1}{2\rho c_H} \int_{x-c_H t}^{x+c_H t} Q\,dx$$

(Fig. 5), which, in the shallow-water approximation, propagate without dispersion with velocity $c_H = (gH)^{1/2}$.[36] In an incompressible liquid the velocity of the flow under the wave of a rise of the level is

$$u = \Delta h c_H H^{-1} = \frac{1}{2\rho H}\left(\int Q\,dx\right)^{-1} = \frac{P_W}{2\rho H}.$$

Disturbances whose scale is the same as that of the envelope and which recede away from the wave packet carry off the momentum

$$P_0 = \int_{-\infty}^{\infty} Hu\,dx = \int_{-\infty}^{\infty} P_W dx. \qquad (2.16)$$

Thus although Stokes transport is completely compensated by the induced counterflow and the mass flux through any section is equal to zero, the total momentum, taking into account the rapidly receding long-wavelength disturbances, exactly equals the quasimomentum (2.16). Obviously, an impulse with exactly the same magnitude (2.16) must be applied in order to excite the wave packet. In this case the momentum and energy of the emitted quasimonochromatic waves can be studied on the basis of the linear theory, ignoring the induced flows.

An analogous picture will also hold in the limit $H \to \infty$. In this case, the flow shown in Fig. 1 has the dipole moment

$$A = \int_{-\infty}^{\infty} xQ\,dx.$$

The momentum of the flow consists of two parts. The first part is determined by the region near the wave packet—this contribution is equal to zero exactly, since here Stokes transport is compensated by the induced counterflow. The second part of the momentum is represented by the contribution from the far zone—it is determined by the propagating long-wavelength disturbances, emitted at the moment the packet arises. In this case, obviously, gravity waves with long wavelength are such disturbances.[33,34]

Other types of disturbances can also carry away momentum. For example, if there is a jump in the density (pycnocline) near the surface, then the appearance of a packet of surface waves is accompanied by the emission of an internal wave on the pycnocline with wavelength of the order of the size of the envelope.[38] If the pycnocline lies at a quite large depth, then the surface waves do not interact directly with the pycnocline and the perturbation of the pycnocline is determined only by the induced flow. The propagating internal waves also carry away some of the momentum. However the total momentum emitted by all long-wavelength modes accompanying the appearance of the wave packet exactly equals the quasimomentum

$$P_0 = \int_{-\infty}^{\infty} P_W dx.$$

2.3. Absorption of a packet of surface waves. The residual vortex

We shall study the changes in the average flow which are brought about by the dissipation of the wave. The simplest example here is a two-dimensional surface wave train. Viscous dissipation of the train in the boundary layer at the free surface leads to the appearance of surface flow.[25] The momentum density associated with Stokes transport transforms into the momentum density of the surface flow after the wave decays. Thus, instead of the system of sources and sinks owing to gradients of the intensity of Stokes transport, there arises a system of surface flows with the same mass flow that was concentrated between the crests and troughs of the waves and with the same sources and sinks owing to the gradients of this flow. As a result, at the location of the dissipated packet, with which the induced dipole counterflow is connected (see Fig. 1), there remains in the volume of the liquid a vortex consisting of precisely the same dipole counterflow and surface flow and assuming the quasimomentum of the wave (Fig. 6).

We note that if the wave packet propagates with the group velocity, then the residual vortex arising after the wave decays moves much more slowly—its velocity is proportional to the intensity of the surface flow. To excite such a vortex by some external action an impulse $\mathbf{P}_0$ would have to be applied. In the process, momentum of the same magnitude would be carried off by fast long-wavelength disturbances.

Thus under conditions of viscous damping of a packet of gravity waves there forms a residual vortex whose momentum is equal to the momentum of the wave packet. The pattern of surface sources and sinks for average flows [the quantity on the right sides in the system (2.8)] does not change in this case, so that the emission of waves with long wavelength and with the scale of the envelope does not occur here.[1)]

Absorption of waves by a thin, freely floating body—a float (Fig. 7)—occurs analogously. The absorbed wave packet gives up its momentum, associated with the Stokes transport (quasimomentum), to the float. As a result the float starts to move along the phase velocity of the wave and acquires a momentum P_0. The weight of the float in hydro-

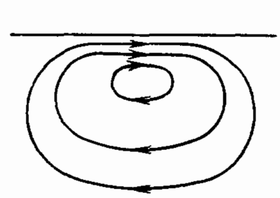

FIG. 6. Residual vortex arising as a result of viscous dissipation of a surface wave.

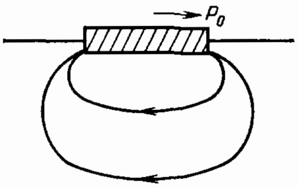

FIG. 7. Motion of a float which has absorbed a surface wave.

static equilibrium is equal to the weight of the liquid displaced by it. Therefore the float creates the same dipole flow pattern and has the same momentum as would the volume of liquid filling the submerged part and moving with the same velocity. Thus the total momentum of the flow is equivalent to the momentum of the vortex in Fig. 6, the difference being that the moving float plays the role of the surface flow. As in the case of the formation of a residual vortex, long-wavelength disturbances are not emitted here. The induced dipole counterflow is transformed in this case into a potential flow around the float. We note that the flow around a body in an incompressible liquid is characterized by some virtual mass[36]; for a thin body this virtual mass is equal to zero, while the total momentum is equal to the momentum P_0 of the float itself.

The absorption of a wave packet by a fixed wave absorber is different. In this case the sources and sinks associated with the gradient of the mass flux vanish, and the induced counterflow vanishes with them. The absorber absorbs the momentum P_0. At the same time, owing to the change in the field of the mass sources, long-wavelength disturbances, which carry off the momentum $-P_0$, are emitted. These disturbances escape, and the neighborhood of the wave absorber remains in a state of complete rest.

Above we studied wave-induced motions in an ideal incompressible liquid. For a wave packet in a continuous medium of a different type the corresponding picture can look different. For example, for an electromagnetic wave in an elastic medium there arises a ponderomotive force that acts on a particle of the medium in the region of the leading and trailing slopes of the wave packet. The result of the action of this force in a solid body is not free flow, as happened in a uniform liquid, but rather a finite displacement of the particles which is proportional to the intensity of the wave. In this case the dissipation of the wave is not accompanied by the formation of a residual vortex, but rather long-wavelength acoustic disturbances are emitted.[39] Another interesting possibility, pointed out in Ref. 39, is Cherenkov emission of sound by a wave packet in a medium, if the group velocity of the packet exceeds the velocity of sound.

Induced flows for internal gravity waves in a stratified medium are an intermediate case. Induced vortex flow arises in the horizontal plane, while vertical motion occurs in the same manner as in an elastic medium: buoyancy forces compensate the action of the ponderomotive forces and a finite vertical displacement of the fluid particles, proportional to the intensity of the wave, arises.[40] The calculation of the emission of long-wavelength disturbances (whose scale is the same as that of the envelope of the wave packet) accompanying generation, dissipation, and scattering of internal waves is of great interest in geophysics (see Ref. 41, where the first step in this direction was taken).

In summarizing the results of this section we conclude that the traditional description of wave disturbances in a medium, based on the application of the concepts of quasienergy, quasimomentum, and wave action (number of quasiparticles), has a definite physical meaning (see also Ref. 42). In most cases the quasienergy and quasimomentum can be regarded as the "true" energy and momentum of the wave packet in the moving medium. This happens, in particular, in the study of different problems involving the absorption and emission of waves as well as in problems involving the nonlinear interaction of waves with low amplitude.[16] In studying below different questions regarding the interaction of waves with flows we shall employ the concept of wave energy precisely in the sense adopted here.

3. SUPER-REFLECTION

Here we shall study different aspects of one of the most effective mechanisms of amplification and absorption of waves in a nonuniformly moving medium. This mechanism is associated with the existence of waves with negative energy or a change in the sign of the dissipation in the hydrodynamic flow. It is convenient to study this mechanism for the example of the simplest hydrodynamic flows—a tangential discontinuity (TD) of the velocity—and other flows with piecewise-constant vorticity, where the effect is manifested in pure form and can be studied analytically. Amplification (super-reflection) was first noted for sound incident on a TD,[43,44] and was then studied for other types of waves: internal gravity,[45] electromagnetic,[46] and others.

3.1. The Miles-Ribner problem

We shall discuss in detail the simplest problem of the reflection of a monochromatic sound wave $\exp(-i\omega t + ikx)$ from a TD (Fig. 8). Joining the solutions for the potential, the pressure p, and the vertical displacement (in the y direction) η of the particles in the medium at rest (1) and in the moving medium (2)

$$\begin{aligned} &\varphi_1 = e^{iq_1y} + Re^{-iq_1y}, \quad p_1 = i\omega\rho\varphi_1, \\ &\eta_1 = -\frac{q_1}{\omega}\left(e^{iq_1y} - Re^{-iq_1y}\right), \\ &\varphi_2 = Te^{iq_2y}, \ p_2 = i(\omega - kU)\rho\varphi_2, \\ &\eta_2 = -\frac{q_2}{\omega - kU}Te^{iq_2y}, \end{aligned} \tag{3.1}$$

we find with the help of the boundary conditions

$$p_1 = p_2|_{y=0}, \ \eta_1 = \eta_2|_{y=0}$$

the coefficients of reflection and refraction of pressure waves:

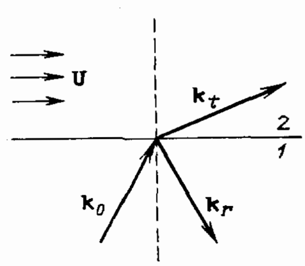

FIG. 8. Reflection and refraction of sound waves at a tangential discontinuity. k_0 is the wave vector of the incident wave; k_r and k_t are the wave vectors of the reflected and trasmitted waves.

$$R = \frac{(q_1/\omega^2) - [q_2/(\omega - kU)^2]}{(q_1/\omega^2) + [q_2/(\omega - kU)^2]}, \quad T = \frac{2q_1/\omega\,(\omega - kU)}{(q_1/\omega^2) + [q_2/(\omega - kU)^2]}, \tag{3.2}$$

where $q_1 = [(\omega^2/c^2) - k^2]^{1/2}$, $q_2 = \{[(\omega - kU)^2/c^2] - k^2\}^{1/2}$, and c is the velocity of sound. The sign of the vertical component of the wave vector in a moving medium is determined by the radiation condition $v_{gry} > 0$ which can be derived from the solution of the starting problem.[47] It follows from the dispersion equation $(\omega - kU)^2 = c^2(k^2 + q_2^2)$ that

$$v_{gry} \equiv \frac{\partial\omega}{\partial q_2} = \frac{c^2 q_2}{\omega - kU}.$$

If $\omega - kU < 0$, the radiation condition requires that we choose the branch $q_2 < 0$. In this case the reflected wave is amplified: $|R| > 1$.

To interpret the super-reflection effect it is necessary to determine the sign of the energy of the refracted wave. Setting the average Eulerian velocity equal to zero $\langle \mathbf{V} \rangle = 0$ (no induced flows) we obtain the average momentum density of a monochromatic sound wave in the medium at rest:

$$\mathbf{P} = \frac{1}{2}\operatorname{Re}\tilde{\rho}\mathbf{v}^* = \frac{\mathbf{k}}{2\rho c^2\omega}|p|^2 = \frac{k}{\omega}\mathscr{E}_0 \tag{3.3}$$

(the asterisk denotes complex conjugation), where $\tilde{\rho} = p/c^2$, p, and $\mathbf{v} = (\mathbf{k}/\omega)p$ are the amplitudes of the oscillations of the density, pressure, and velocity, calculated on the basis of the linear problem, and

$$\mathscr{E}_0 = \frac{1}{2}\left(\frac{\rho|\mathbf{v}|^2}{2} + \frac{|p|^2}{2\rho c^2}\right) = \frac{|p|^2}{2\rho c^2}$$

is the average energy density in the medium at rest (the fact that the average kinetic energy in the sound wave is equal to the average potential energy in the sound wave is taken into account).[36] We note that the momentum here, as for surface waves, is related with the drift of the particles (the average Lagrangian velocity):

$$\mathbf{P} = \rho\langle \mathbf{V}_Л \rangle = \frac{1}{2}\rho\operatorname{Re}[(\zeta,\ i,\ \mathbf{k})\mathbf{v}] = \frac{\mathbf{k}}{\omega}\frac{|p|^2}{2\rho c^2}.$$

In a moving medium, since the amplitude of the pressure does not depend on the coordinate system, we obtain from (2.2) the following expression for the average energy density:

$$\mathscr{E} = \mathscr{E}_0\frac{\omega}{\omega - kU} = \frac{\omega|p|^2}{2\rho c^2(\omega - kU)}. \tag{3.4}$$

For $\omega - kU < 0$ the energy density is negative. The vertical component of the energy flux density is given by

$$S_y = v_{gry}\mathscr{E} = \frac{q_2\omega|p|^2}{2\rho(\omega - kU)^2} = \frac{\rho\omega}{2}q_2|\varphi|^2.$$

For $q_2 < 0$ the energy flux of the refracted wave is oriented toward the discontinuity: $S_y < 0$. Thus amplification occurs owing to the influx of energy from the moving medium. In the process a negative-energy wave is emitted into the medium. The law of conservation of energy $q_1(1 - |R|^2) = q_2|T|^2$ can be checked directly from the expressions (3.2).

We shall study different regimes of reflection depending on the angle of incidence θ and Mach's number $\mathrm{Ma} = U/c$. Representing the wave vector of the incident wave in the form $\mathbf{k}_0 = (\omega/c)(\sin\theta, \cos\theta)$, we rewrite the reflection coefficient (3.2) in the form

$$R = \frac{\cos\theta - [1 - \sin^2\theta\,(1 - \mathrm{Ma}\sin\theta)^{-2}]^{1/2}}{\cos\theta + [1 - \sin^2\theta\,(1 - \mathrm{Ma}\sin\theta)^{-2}]^{1/2}}. \tag{3.5}$$

Three different reflection regimes are possible (Fig. 9):

1) normal reflection ($q_2 > 0, |R| < 1$) with $\sin\theta < (\mathrm{Ma} + 1)^{-1}$;

2) total reflection ($\operatorname{Re} q_2 = 0$, $|R| = 1$) with $(\mathrm{Ma} + 1)^{-1} \leqslant \sin\theta \leqslant 1$, if $\mathrm{Ma} \leqslant 2$, and with $(\mathrm{Ma} + 1)^{-1} \leqslant \sin\theta \leqslant (\mathrm{Ma} - 1)^{-1}$, if $\mathrm{Ma} > 2$; and,

3) Super-reflection ($q_2 < 0, |R| > 1$) with $\mathrm{Ma} > 2$ and $\sin\theta > (\mathrm{Ma} - 1)^{-1}$.

In the last case there exists a resonant angle of incidence

$$\theta_0 = \arcsin\frac{2}{\mathrm{Ma}},$$

for which $|R| = \infty$. Spontaneous Cherenkov emission of a vortex sheet—an infinitely thin layer of liquid moving with velocity $U/2$, in which the vorticity $\mathbf{\Omega} = \operatorname{rot}\mathbf{v}$ does not vanish[43,44]—occurs. In the process of spontaneous emission a negative-energy wave is emitted into the moving medium and a positive-energy wave is emitted into the medium at rest. The oscillations of the TD do not decay and they are not amplified, and the energy of the sound emitted into the medium at rest is taken from the entire moving medium.

For

$$\mathrm{Ma} \geqslant 2\sqrt{2}$$

the TD becomes stable,[36] and unstable Kelvin-Helmholtz surface modes are transformed into waves with the phase velocity (relative to the vortex sheet)

$$V = \pm c\left[1 + \frac{\mathrm{Ma}^2}{4} - (1 + \mathrm{Ma}^2)^{1/2}\right],$$

traveling along the discontinuity. Since

$$V + \frac{U}{2} > c,$$

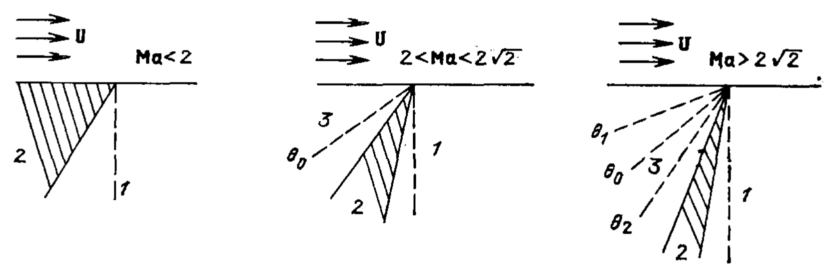

FIG. 9. Different regimes of reflection of sound from a tangential discontinuity. The ranges of the angles of incidence are indicated. 1—Normal reflection, 2—total reflection, 3—superreflection.

Vavilov-Cherenkov radiation should appear. Disturbances traveling along the TD emit sound at angles $\theta_{1,2}$, for which

$$\sin\theta_{1,2}=\left\{\frac{\mathrm{Ma}}{2}\pm\left[1+\frac{\mathrm{Ma}^2}{4}-(1-\mathrm{Ma}^2)^{1/2}\right]^{1/2}\right\}^{-1} \quad (3.6)$$

In a compressible gas with

$$\mathrm{Ma}\geqslant 2\sqrt{2}$$

a TD is stabilized only within the framework of a two-dimensional model. For disturbances traveling at an angle to the plane, shown in Fig. 8, the projection of the velocity of the medium on the direction of the wave vector, determining the action of the motion of the medium on the propagation of waves (i.e., actually the Doppler shift), decreases. As a result the condition of stability of such disturbances has the form

$$\mathrm{Ma}\cos\theta\geqslant 2\sqrt{2}.$$

Thus disturbances traveling at a quite large angle θ are always unstable—complete stabilization of a supersonic TD in a gas is impossible.[48] Taking into account the fact that for $\theta=\pi/2$ the wave no longer "feels" the motion of the medium and the instability vanishes we arrive at the conclusion that there exist maximally unstable disturbances for which the quantity $\mathrm{Im}(\omega/k)$ and the corresponding angle $\theta_{\max}$ depend on the parameter Ma.[49]

We note, however, that the solutions obtained on the basis of two-dimensional hydrodynamics can be realized for waves on shallow water.[36] Here, in particular, the stabilization of TD examined above becomes possible[50] if

$$\mathrm{Ma}\equiv\frac{U}{(gH)^{1/2}}>2\sqrt{2}.$$

The existence of negative-energy waves in a supersonic TD can also lead to different dissipative instabilities. For example, by placing into a moving medium a boundary that reflects sound we add acoustic feedback to the TD which intensifies the reflected sound.[51] The instability arising in this case can be interpreted as a dissipative instability of the modes of the waveguide formed by the TD and the reflecting boundary. These modes have negative energy, and dissipation occurs owing to the emission of sound into the medium at rest. An instability of this type was observed in supersonic boundary layers.[1]

The "spreading" of the TD, i.e., the replacement of the TD with a transitional layer of finite thickness, also leads to instability.[52] The poles of the reflection coefficient $R(\omega,k)$ move away from the real axis ω into the complex plane.[53] In addition, the neutrally stable characteristic modes of the TD, accompanied by spontaneous emission of sound, transform into unstable modes.

If the TD is unstable, then the problem of the reflection of a plane monochromatic wave may turn out to be improperly posed. The validity of the solutions obtained can be studied on the basis of more complicated models, which take into account the non-steady-state and nonuniform nature of the wave field, the finite width of the shear layer, etc. We shall examine below some results for sources near a TD which are bounded in space and time; these results permit evaluating the applicability of Miles' solution for plane waves.

3.2. Excitation of a tangential discontinuity by an incident wave

In studying plane monochromatic waves we assumed that the sound and the characteristic oscillations of the discontinuity are linearly independent modes. At the same time sound waves from a real source having finite dimensions and a finite duration initiate the Kelvin-Helmholtz instability on the TD. The solution of this problem presents certain difficulties of both a technical character and in giving a physical interpretation of the results.[54] Difficulties even arise in the simplest formulation of the problem of reflection of a plane pulse from the interface of equilibrium media at rest, since precursors, which are inconsistent with causality considerations, formally arise in the reflected field. The problem of eliminating precursors is studied in detail in Ref. 54, where it is shown that precursors are formed only if the intersection of the incident wavefront with the interface separating the media lasts for an infinitely long time. Big difficulties also arise in the solution of problems concerning the reflection of a wave from nonequilibrium media, an example of which are TDs; here there arise problems in choosing the structure of the wavefront of the transmitted wave, in describing the shape of the reflected signal, etc.

All these difficulties can be avoided by using Laplace's method to solve the indicated problems formulated as initial-value problems.[55,56]

We shall study as an example the emission from a monochromatic point source of mass with unit strength at a distance h from the TD.[56] The wave equations in the medium at rest and in the moving media have the following forms, respectively,

$$\Delta\varphi_1+\left(\frac{\omega}{c}\right)^2\varphi_1=\delta(x,y+h),$$

$$\Delta\varphi_2-\frac{1}{c^2}\left(-i\omega+U\frac{\partial}{\partial x}\right)^2\varphi_2=0. \quad (3.7)$$

Taking into account the boundary conditions at the TD it is not difficult to obtain the solution by the method of Fourier transformation in the coordinate x:

$$\varphi=\varphi_0+\frac{1}{2\pi}\int\frac{iR(\omega,k)}{2q_1}\exp\left[ikx-iq_2(y-h)\right]dk, \quad (3.8)$$

where φ_0 is the field of the source in an unbounded medium at rest and $R(\omega,k)$ is determined by the expression (3.2).

The contour of integration in the k plane must be chosen based on the principle of causality. Since we want to obtain the solution of the initial-value problem by Laplace's method we must study the complex values of ω corresponding to growing waves, i.e., values of ω quite far up in the upper half of the ω plane. The poles of the reflection coefficient $R(\omega,k)$ lie in the upper half of the complex k plane. The integration can then be performed over the real axis k. In order to continue analytically the obtained solution to real values of ω the integration path must be deformed in the complex k plane, adding to the real axis loops around the poles k_i in the lower half-plane and arcs around the poles k_s on the real axis (Fig. 10).

The poles k_i correspond to the characteristic oscillations of the discontinuity which grow along the x axis. Thus the complete solution of the problem of a point source includes not only traveling sound waves [obtained by integra-

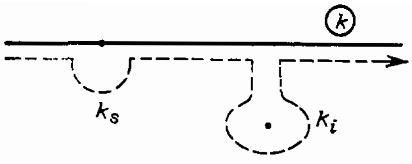

FIG. 10. Contour of integration in the complex k plane.

tion along the real axis in (3.8)], but also a surface wave growing along the x axis on the discontinuity.

The solution found in this manner permits establishing the limits of applicability of the results concerning reflection of monochromatic plane waves from an unstable TD. Indeed, as the point source recedes to infinity ($h\to\infty$), the incident cylindrical wave approaches a plane wave near a fixed direction. The efficiency of excitation of the surface wave decreases exponentially. We also note that the lines of constant amplitude of the surface wave are rays with the slope $\tan\theta_c = \mathrm{Im}\,q/\mathrm{Im}\,k = 1$ ($\theta_c = 45°$).[56] For angles of incidence $\theta > \theta_c$ the solution in the form of refracted and reflected waves becomes meaningless, since it exists against the background of an exponentially growing solution of the surface-wave type. At the same time for $\theta < \theta_c$ the surface wave can be neglected.

Interesting features arise when the TD is excited by a non-steady-state source.[57] Here the model of a source in the form of an instantaneous point impulse $\delta(t)\delta(\mathbf{r})$ is not applicable. In this case the solution of the initial-value problem by Laplace's method becomes much more complicated: because of the existence of an increment of instability of characteristic oscillations of the TD that increases without bound as the wave number increases the singular points in the integrand recede to infinity along the imaginary axis in the complex ω plane. For this reason it is not possible to choose a contour in the ω plane that would pass above all singularities. The reflection of sound pulses from a TD was studied in Ref. 57. It was shown that for a stable TD ($\mathrm{Ma} < 2\sqrt{2}$) the sound field is a superposition of the reflected and lateral waves, as well as three characteristic waves emitted by the discontinuity at angles θ_0, θ_1, and θ_2. For an unstable TD the incident sound pulse excites, together with lateral and reflected waves, a region of growing disturbances which expands along the TD; this leads, already in the linear approximation, to essentially explosive (i.e., over a finite time) decomposition of the TD. The region of instability in space has the form of a triangle, whose shape depends strongly on the spatial spectrum of the source (this phenomenon is called configurational instability).

The characteristics of configurational instability of a TD in an incompressible medium, taking into account gravity and surface tension, are analyzed in Ref. 58.

The expression for the sound field, excited by a source near a TD, permits finding an important characteristic of the source—its acoustic impedance (the ratio of the amplitude of the sound pressure at the source and its strength[59]). In particular, the solution (3.8) permits calculating the acoustic radiation resistance of a point source near the TD: $r_a = -\mathrm{Im}[\rho\omega\varphi(x=0, y=-h)]$. For subsonic discontinuities, in the limit $\mathrm{Ma}\to 0$, we have $q_{1,2} = ik$, and the reflection coefficient has a simple form:

$$R=[(\omega-kU)^2-\omega^2][(\omega-kU)^2+\omega^2]^{-1}.$$

In this case the integral along the real axis in (3.8) obviously does not contribute to r_a. The radiation resistance is determined by the excited surface wave, which corresponds to the integral over the contour around the pole $k i = (1-i)\omega/U$. As a result we have

$$r_a = -\frac{1}{4}\rho\omega\sin\left(\frac{2\omega h}{U}+\frac{\pi}{4}\right)\exp\left(-\frac{2\omega h}{U}\right). \tag{3.9}$$

The quantity r_a depends on the parameter $2\omega h/U$ and can change sign. This last fact makes it possible to explain the mechanism of self-excitation of some types of whistles.[20,60] In reality, if the source of mass is, for example, a Helmholtz resonator, then for $r_a < 0$ the oscillations in the resonator will be amplified.

It should be emphasized that for self-excitation of a resonator it is not so much the instability of the flow that is important, but rather the presence of characteristic oscillations of the flow, its inertial properties. In particular, the source can have a negative radiation resistance; it can even excite neutrally stable oscillations in the flow owing to the development of such oscillations as the jet passes through the region where efficient interaction with the source occurs.[60] The characteristic size of this region—the "transit length"—in the example presented above is determined by the distance h from the source to the TD. This mechanism is analogous to the mechanism of self-excitation of electronic microwave devices, where the development of the perturbations in the electron beam over a transit length h is determined by the parameter $\omega h/U$ and leads to bunching of electrons.[61] In a hydrodynamic flow an external impulse (from the acoustic resonator) also leads to an effect similar to bunching: in the process of the development of the fundamental mode of the oscillations of the flow the velocity perturbations transform into pressure perturbations. The latter perturbations, in their turn, excite the resonator, thereby closing the feedback loop. We note here that in those cases when the characteristic modes of the electron beam are unstable the excitation increment of the electromagnetic resonator is determined, as in hydrodynamics, by the integral of the wave number in the complex plane over a contour around poles corresponding to waves that grow downstream.[62]

3.3. Negative dissipation in a moving medium

Discontinuities in a viscous medium spread into shear layers of finite thickness, within which small disturbances have a complicated structure.[1] The effect of viscous dissipation on the reflection of sound from the moving medium can, however, be evaluated by remaining within the framework of the TD model, as done in Refs. 63 and 64. For this we shall examine the discontinuity between a viscous and an ideal liquid, i.e., we shall take into account the viscosity only in the moving layer for $y > 0$. It should be emphasized that this simple model with a discontinuity of the viscosity is suitable only for qualitative calculations and can be employed to study certain physical effects, for example, the effect of weak dissipation in a moving medium on the propagation of sound with a given amplitude at the boundary. For other types of problems, in particular, in the study of the stability of small disturbances, such a model may turn out to be inapplicable.

Disturbances of the velocity in a moving viscous medium can be represented in the form

$$u=\frac{\partial\varphi}{\partial x}-\frac{\partial\psi}{\partial y},\quad v=\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial x}.$$

The amplitudes of the potential φ and the stream function ψ satisfy the equations[63]

$$\frac{d^2\varphi}{dy^2}+q_2^2\varphi=0,\quad \frac{d^2\psi}{dy^2}+q_1^2\psi=0, \tag{3.10}$$

where

$$q_1=[i(\omega-kU)\nu^{-1}-k^2]^{1/2},$$

$$q_2=\left\{\frac{(\omega-kU)^2}{c^2}\left[1-\frac{4i\nu}{3c^2}(\omega-kU)\right]^{-1}-k^2\right\}^{1/2},$$

while the pressure amplitude is given by

$$P=i(\omega-kU)\rho\varphi\left[1-\frac{4i\nu}{3c^2}(\omega-kU)\right]^{-1}. \tag{3.11}$$

Thus the general solution

$$u=ik\varphi_0e^{iq_2y}-iq_1\psi_0e^{iq_1y},$$
$$v=iq_2\varphi_0e^{iq_2y}+ik\psi_0e^{iq_1y} \tag{3.12}$$

is a superposition of two modes: a potential mode, representing the modified viscosity of the sound wave, and a vortex mode, for which there are no pressure oscillations in the linear approximation. For large Reynolds numbers $\mathrm{Re}=\nu|\omega-kU|/c^2$ the vortex viscous mode oscillates rapidly and decays as $y\to\infty$.

The dynamic boundary conditions at the TD are that the components of the momentum flux normal to the discontinuity, i.e., the components

$$\sigma_{xy}=\rho\nu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$
$$\sigma_{yy}=-P+2\rho\nu\frac{\partial v}{\partial y}-\frac{2}{3}\rho\nu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \tag{3.13}$$

of the stress tensor, must be continuous.[36] These boundary conditions, together with the kinematic condition[2)] (continuity of the displacement $\eta=iv(\omega-kU)^{-1}$ permit joining the viscous solution (3.11) and (3.12) for $y>0$ and the solution (3.1) for φ_1, and p_1 in an ideal fluid at rest for $\mathrm{Re}^{-1}\ll 1$. As a result we obtain the reflection coefficient for sound:

$$R=\frac{[q_1(\omega-kU)^2/q_2\omega^2]-1+[4i\nu k^2/(\omega-kU)]}{[q_1(\omega-kU)^2/q_2\omega^2]+1-[4i\nu k^2/(\omega-kU)]}. \tag{3.14}$$

As $\nu\to 0$ the expression (3.14) transforms into (3.2). In the range of total reflection, where for $\nu=0$ the reflection coefficient is $|R|=1$, the correction owing to viscosity determines the magnitude of the relative dissipation:

$$1-|R|^2=16\nu k^2(\omega-kU)\frac{q_1|q_2|\omega^2}{q_1^2(\omega-kU)^4+|q_2|^2\omega^4}. \tag{3.15}$$

For $\omega-kU<0$ the dissipation in the moving medium becomes negative, which leads to amplification of the reflected wave: $|R|>1$.

4. WAVES ON A TANGENTIAL DISCONTINUITY

The mechanism of interaction of waves whose energies have opposite signs and the concomitant diverse effects can be traced by studying the interaction of different types of waves with the TD. We note that this is the most convenient model of flow for studying a wide class of hydrodynamic phenomena in different media: the atmosphere and the ocean, moving plasma, many astrophysical objects, etc.

4.1. The interface between a heavy and a light liquid

Waves which occur on the surface of the interface between media with different density and which interact with a TD are the classical object studied in the theory of hydrodynamic instability. We call attention, in particular, to the large number of works on the Kelvin-Helmholtz instability (see, for example, the review of Ref. 65). Here we shall study only some relatively new and as yet little known results.

4.1.1. *Amplification of surface gravity waves on reflection from a tangential discontinuity.* The super-reflection of surface waves incident on a TD in deep water (Fig. 11) is of great interest in the physics of the ocean. The corresponding boundary-value problem reduces to solving a system of integral equations; great mathematical difficulties are encountered in constructing the solution.[66] An approximate solution of this problem is found in Ref. 67. Galerkin's method was used to derive an expression for the reflection coefficient for the amplitude of deflections of a free surface:

$$R=\frac{(k_{y1}/k_{y2})N^2-1-iX}{(k_{y1}/k_{y2})N^2+1-iX}, \tag{4.1}$$

where

$$k_{y1}=\frac{\omega^2}{g}\cos\theta,$$

$$k_{y2}=\frac{\omega^2}{g}(1-\mathrm{Ma}\sin\theta)\left[(1-\sin\theta)^2-\frac{\sin^2\theta_1}{(1-\mathrm{Ma}\sin\theta_1)^2}\right]^{1/2}$$

are the projections of the wave vectors for the incident and reflected wave, N and X are real functions of the angle of incidence θ_1 and angle of refraction θ_2 as well as the ratio of the jump in the velocity to the phase velocity of the gravity wave $\mathrm{Ma}=\omega U/g$, analogous to the Mach number in acoustics.

The asymptotic behavior of the refracted field in the limit $y\to\infty$ and the reflection regime are determined by the wave number k_{y_2}. For

$$\mathrm{Ma}<\mathrm{Ma}_1\equiv\frac{1-(\sin\theta_1)^{1/2}}{\sin\theta_1}$$

normal reflection occurs: $k_y^2>0$ and $|R|<1$. In the region

$$\mathrm{Ma}_1<\mathrm{Ma}<\mathrm{Ma}_2\equiv\frac{1+(\sin\theta)^{1/2}}{\sin\theta_1}$$

total reflection occurs: $k_{y_2}=0$, $|R|=1$. Finally, for $\mathrm{Ma}>\mathrm{Ma}_2$ super-reflection occurs, when $k_{y_2}<0$ and $|R|>1$.

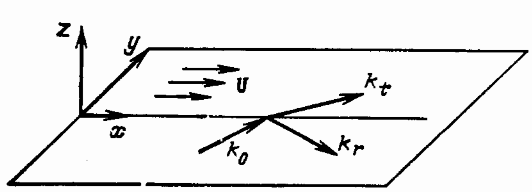

FIG. 11. Reflection and refraction of surface waves on deep water incident on a tangential discontinuity.

In this case for the refracted wave we have

$$\omega - kU = \omega(1 - \mathrm{Ma}\sin\theta_1) < 0$$

and the energy of the wave is negative. The phase velocity of the reflected wave is oriented toward the discontinuity, while the wave packet moves away from the discontinuity in the direction of the group velocity, whose corresponding projection is

$$v_{gry}^{(2)} = \frac{g^2 k_{y2}}{2(\omega - kU)^3} > 0.$$

In the case when $\sin\theta_1 = 2/\mathrm{Ma}(|\omega - kU| = \omega)$ a resonance of the incident wave with the characteristic mode of the discontinuity arises; here $|R| = \infty$. A single characteristic mode with frequency ω exists for $\mathrm{Ma} > 2$ and represents an inhomogeneous plane wave whose wave vector makes the angle

$$\gamma = \frac{\pi}{2} - \theta_1 = \arccos\frac{2}{\mathrm{Ma}}$$

with the discontinuity. The group velocity in it is oriented differently on different sides of the discontinuity. This mode represents essentially Cherenkov emission of vortex-sheet disturbances.

Super-reflection of surface waves can lead, in particular, to instability of jet flows relative to disturbances of the free surface.[67]

4.1.2. *Instability of a tangential discontinuity in a stratified medium.* We shall now examine the discontinuity of the density $\Delta\rho = \rho_0 - \rho_1$, which is a TD, in the presence of gravity g and surface tension σ. We shall assume that $a \equiv \rho_1/\rho_0 \ll 1$, keeping in mind the case of an interface between water and air ($\rho_1/\rho_0 \approx 10^{-3}$) which is important for practical applications. We shall follow the evolution of the dispersion curve $\omega(k)$ of surface waves on the discontinuity as a function of the jump in the velocity U

$$\omega = \frac{a}{a+1}kU \pm \left[\frac{1-a}{1+a}gk + \frac{\sigma k^3}{\rho_0(1+a)} - \frac{a}{(1+a)^2}k^2U^2\right]^{1/2}, \tag{4.2}$$

in a system of coordinates in which the bottom heavy liquid is at rest.[36,68] For $U = 0$ the expression (4.2) characterizes the standard gravity-capillary waves traveling in opposite directions (Fig. 12, curve 1). For $U \neq 0$ the dispersion curves become unsymmetric owing to the fact that the waves are carried off by the flow. For

$$U > U_c = \left(4\frac{1-a}{a^2}\frac{g\sigma}{\rho_0}\right)^{1/4}$$

a section where the frequency changes sign appears on the bottom branch of the dispersion curve (see Fig. 12, curve 2): $\omega - kU < 0$. This section corresponds to a wave with negative energy. As U is further increased both branches continue to converge, and finally when

$$U = U_{KH} = U_c(1+a)^{1/2},$$

they reconnect (Fig. 12, curve 3); a Kelvin-Helmholtz instability arises. Thus the appearance of this instability can be attributed to the interaction of waves whose energies have different signs.[68,69]

Negative-energy waves can grow not only owing to interaction with positive-energy waves, but also as a result of some other mechanisms for extracting energy. Thus, analogously to the plasma dissipative instability,[70] taking into account viscosity in a layer at rest leads to growth of negative-energy waves.[68] Such waves become damped, if the top moving layer is viscous. In this case, however, positive-energy waves can grow on the top branch of the dispersion curve, since the viscosity in the moving medium with

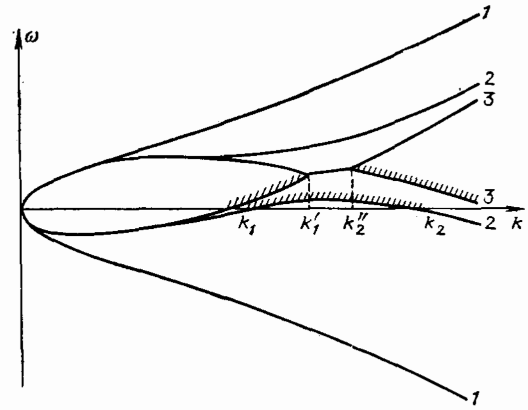

FIG. 12. Dispersion curves for the Kelvin–Helmholtz model with surface tension: $U = 0$ (1), $U_c < U < U_{KH}$ (2), and $U > U_{KH}$ (3). The shaded sections correspond to waves with negative energy in a coordinate system that is stationary relative to the bottom layer of the liquid.

$$U > U_c' = [(1-a)g\sigma\rho_0^{-1}]^{1/2}$$

leads to negative damping of waves for which $\omega - kU < 0$. Indeed, in transferring to a coordinate system where the top layer is at rest the energy of the growing mode changes sign and at the same time the dissipation changes sign[68]: in this coordinate system the negative-energy wave grows under the action of positive dissipation. In the process, evidently, the presence of the instability at this mode is invariant to the choice of coordinate system.

The appearance of dissipative instability can be conveniently illustrated in a system moving with the "mean-mass" velocity

$$V = \frac{a}{1+a}U.$$

In this coordinate system the light liquid moves in the positive direction with velocity $U/(1+a)$, while the heavy liquid moves in the opposite direction with the lower velocity

$$U\frac{a}{1+a}.$$

The branches of the dispersion curve here become symmetric relative to the k axis (Fig. 13). For a sufficiently large jump in the velocity there appear on the dispersion curves sections where the phase velocity of the disturbances is lower than the velocity of the medium; in Fig. 13 they are marked off by rays with slope $U/(1+a)$ (for the light medium on the top) and $-Ua(1+a)^{-1}$ (for the heavy medium on the bottom). The presence of waves for which $\omega - kU < 0$ and which are retarded relative to the moving medium can lead to dissipative instability.

It is obvious that the dissipation changes sign first in the light medium. At the boundary between air and water this

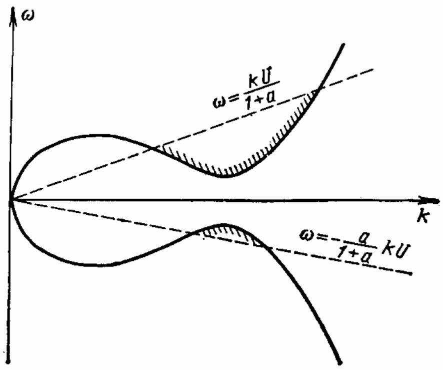

FIG. 13. Dispersion curves for waves on a TD in the "mean-mass" coordinate system. The shaded sections correspond to potentially unstable surface waves.

occurs when the wind velocity exceeds the minimum velocity of gravity-capillary waves.

Dissipation in the heavy medium (water), however, remains positive up to velocities $U = U_c \gg U'_c$. This delays the onset of instability of wind-driven waves by increasing the threshold wind velocity (see Ref. 25).

As the velocity is further increased dissipation also changes sign in the bottom (heavy) layer (for $U > U_c$ see Fig. 12, curve 2), and then at almost the same time for $U > U_{KH}$ ($\approx U_c$, if $a \ll 1$) reconnection of the dispersion branches occurs and the Kelvin-Helmholtz instability arises (see Fig. 12, curve 3). For a water-air boundary this occurs at very significant wind velocities: $U_{KH} \approx 6.5$ m/s. Under real conditions the onset of the wind instability of surface waves occurs much earlier at $U \approx 1.3$ m/s.[25]

It follows from what was said above that for $\rho_1 \ll \rho_0$ the bottom layer of heavy liquid "carries" surface waves, whose dispersion properties (for $U \ll U_{KH}$) are disturbed only slightly under the action of the top layer of a light medium. At the same time, it is precisely the dissipation in air moving faster than the wave and changing sign that gives rise to the development of instability of surface waves, i.e., the appearance of wind-generated waves. The development of this instability can be evaluated on the basis of the simplest model, consisting of a boundary between an ideal liquid and a viscous (light) medium. The increment of the instability here has the form[71] (under the assumption that $\mathrm{Im}\,\omega \ll \mathrm{Re}\,\omega$)

$$\gamma = -2\nu k^2 \frac{a}{1+a} \frac{\omega - \mathrm{kU}}{\omega}. \tag{4.3}$$

We emphasize, however, that the change in the sign of the dissipation with $\omega - kU < 0$ is not related with its specific physical mechanism. In particular, the wind instability of surface waves could be due to different factors, among which the viscosity (possibly, turbulent) in the wind-generated flow can play an appreciable role (see Ref. 72), together with the mechanism of Miles' instability and other well-known mechanisms.[25]

The instability associated with negative dissipation leads to amplification of waves in a wide range of angles, when the Kelvin-Helmholtz instability does not yet occur. This permits distinguishing these two instabilities experimentally. Unfortunately hydrophysical experiments with the participation of negative-energy waves are largely unknown. Leaving aside the numerous works on the excitation of Tolmin-Shlikhting waves in boundary layers, which, according to Benjamin,[73] also have negative energy, we call attention only to the laboratory experiment of Ref. 74, where the development of an instability which can be interpreted as instability of waves with negative energy was observed together with the Kelvin-Helmholtz instability at the interface between two layers of liquid in relative motion.[75]

4.2. Internal gravity waves

Buoyancy waves—internal gravity waves (IGW) in a stratified medium—are apparently the most important examples of the interaction of waves with hydrodynamic flows in geophysical applications,[26,76] since their phase velocity is often comparable to the velocity of real flows in the atmosphere and the ocean.

The equations for two-dimensional oscillations of a layered incompressible medium have the form[26]

$$\begin{gathered}\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{g\tilde{\rho}}{\rho} = 0,\\ \frac{\partial \tilde{\rho}}{\partial t} + v\frac{d\rho}{dy} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.\end{gathered} \tag{4.4}$$

For IGW, as a rule, the Boussinesq approximation corresponding to the limit $d\rho/dy \to 0$, $g \to \infty$, $N = [(-g/\rho)d\rho/dy]^{1/2} = \mathrm{const}$ can be employed. In this case the change in the density on the scale of a wavelength becomes insignificant, but the returning buoyancy force remains. In this approximation the derivative $d\rho/dy$ enters only in the Brunt-Väisälä frequency N, and we can set $\rho = \mathrm{const}$ in the coefficients of the equations. Then we obtain from (4.4) a dispersion equation for IGW and their group velocity $\mathbf{v}_{gr}$:

$$\begin{aligned}\omega^2 &= \frac{N^2k^2}{k^2+q^2} = N^2\sin^2\theta,\\ \mathbf{v}_{gr} &= \frac{Nq}{(k^2+q^2)^{3/2}}(q, -k) = \frac{N\cos\theta}{k_0}(\cos\theta, -\sin\theta),\\ \mathbf{k}\mathbf{v}_{gr} &= 0, \quad \mathbf{k} = k_0(\sin\theta, \cos\theta).\end{aligned} \tag{4.5}$$

We note that only IGW with $\omega \leqslant N$ can propagate.

We shall study the reflection of IGW from TD, separating a medium with different stratification: $N(y<0) = N_1$, $N(y>0) = N_2$. Joining the solution $p_1 = \exp(iq_1y) + R\exp(-iq_1y)$ in the medium at rest for $y<0$ with the solution $p_2 = T\exp(iq_2y)$ in a medium moving with velocity U for $y>0$ we obtain with the help of the boundary conditions at the TD

$$\begin{aligned}R &= \frac{q_1(N_1^2-\omega^2)^{-1} - q_2[N_2^2-(\omega-kU)^2]^{-1}}{q_1(N_1^2-\omega^2)^{-1} + q_2[N_2^2-(\omega-kU)^2]^{-1}},\\ T &= \frac{2q_1(N_1^2-\omega^2)^{-1}}{q_1(N_1^2-\omega^2)^{-1} + q_2[N_2^2-(\omega-kU)^2]^{-1}},\end{aligned} \tag{4.6}$$

where

$$q_1 = -\frac{k}{\omega}(N_1^2-\omega^2)^{1/2}, \quad q_2 = -\frac{k}{\omega-kU}[N_2^2-(\omega-kU)^2]^{1/2}$$

(the fact that for IGWs $q\cdot v_{gry} < 0$ is taken into account). The law of conservation of energy following from (4.6)

$$q_1(N_1^2-\omega^2)^{-1}(1-|R|^2) = q_2[N_2^2-(\omega-kU)^2]^{-1}|T|^2$$

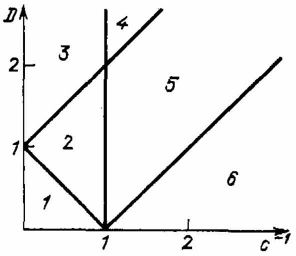

FIG. 14. Separation of the parameter plane, for parameters characterizing the TD in a stratified medium, into regions with different regimes of reflection of IGWs at a discontinuity.

can also be easily derived from the expression (4.5) for $\mathbf{v}_{gr}$ and the formula for the energy density in a IGW in a medium at rest[26]

$$\mathscr{E}=\frac{\rho\langle u^2+v^2\rangle}{2}+\frac{N^2\langle\eta^2\rangle}{2}=\frac{N^2k^2}{\omega^2(N^2-\omega^2)}|p|^2 \qquad (4.7)$$

using the formula (2.2) and the relation $S_y = v_{gry}\mathscr{E}$.

The condition for emission in a moving medium

$$v_{gry} = -\frac{N^2k^2}{(k^2+q^2)^2}\frac{q_2}{\omega-kU}>0$$

makes it necessary to choose for $\omega - kU<0$ the branch $q_2<0$. Since $q_1<0$, we obtain in this case $|R|>1$, i.e., superreflection occurs. The refracted wave, as follows from (2.2), carries off negative energy in the process.

It is convenient to classify the different regimes of reflection in terms of two dimensionless parameters:

$$D=\frac{N_2}{N_1},$$
$$c=\frac{\omega}{|k|U}=\frac{N_1}{k_0U},$$

which determine the character of the refracted field of IGWs.[77] Figure 14 shows the separation of the plane of parameters $(D,\ 1/c)$ into regions for each of which the regimes of reflection of IGW as a function of the angle of incidence θ are shown schematically in Fig. 15. The critical angles $\tilde{\theta}_{1,2}$, bounding the range of total internal reflection, in which there is no transmitted wave, are given by the expressions

$$\sin\tilde{\theta}_1=\frac{Dc}{c-1},\quad \sin\tilde{\theta}_2=-\frac{Dc}{1+c}.$$

Resonances of the TD ($|R|=\infty$), which can be found quite simply in the case of uniform stratification $N_1=N_2=\text{const}$, are of special interest. They are possible only for $c<1$. One of them is determined by the condition $c=1/2(\omega=kU/2)$ and corresponds to the Cherenkov emission of IGWs by a vortex sheet. The angle of emission can be expressed by the relation

$$\sin\theta=\frac{\omega}{N_1}=\frac{kU}{2N_1}.$$

The other two resonances are determined by the condition

$$\operatorname{ctg}\theta=\frac{1-c}{c},$$

which corresponds to the dispersion equation $\omega^2+(\omega-kU)^2=N_1^2$ for the characteristic waves on a TD. These modes are stable for $|k|<\sqrt{2}N_1/U$ and transform into the Kelvin-Helmholtz modes in the limit $N_1\to 0$.

Radiating instability. When a jump in the density is present in a stratified medium the mechanism of dissipation associated with emission of IGWs into the surrounding medium is possible for surface waves at the discontinuity. In a moving medium the energy of the emitted leaky waves[78] can become negative, which will lead to negative dissipation of surface waves, as a result of which such waves grow.[79] The instability associated with radiation losses also occurs in other physical systems: for beams of charged particles in a plasma,[80] in inverted media,[81] and others.

We shall show that wind waves can be generated by the radiating instability mechanism when IGWs are emitted into a stratified atmosphere. Confining our attention to the simplest model, studied in Sec. 4.1, of a TD representing a jump in the density (for a $\ll 1$) we shall take into account the stable stratification of air: $N=N_2$ for $y>0$ and $N=0$ for $y<0$. The dispersion equation for waves in such a system has the form

$$a(\omega-kU)^2\varkappa_2+\omega^2=(1-a)gk+\frac{\sigma}{\rho_1}k^3, \qquad (4.8)$$

where

$$\varkappa_2=\frac{iq_2}{k}=[1-N_2^2(\omega-kU)^2]^{1/2},$$

and in addition the branch of the expression for $\varkappa_2$ is chosen based on the boundary condition in the limit $y\to\infty$. It is not difficult to obtain the instability increment from (4.8) to a first approximation in the small parameter a $\ll 1$:

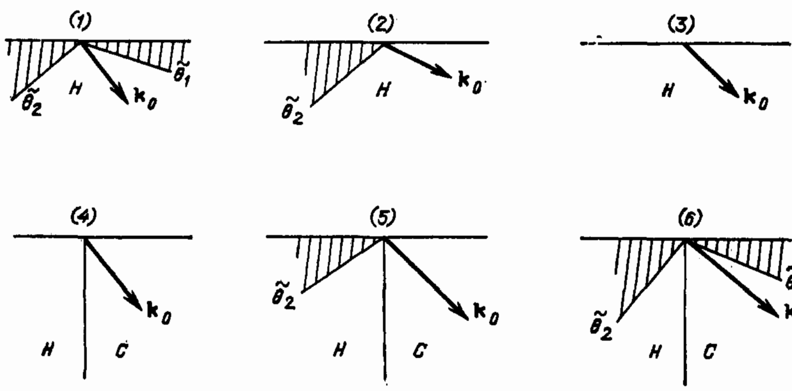

FIG. 15. Regimes of reflection of IGWs. The numbers on the diagrams correspond to the number of the region in the parameter plane D, c^{-1}. The different ranges of angles of incidence, determined by the wave vectors of the incident wave, are singled out: total internal reflection (hatched region), normal reflection (N), and superreflection (S).

$$\gamma = \frac{a}{2}\frac{\omega - kU}{\omega}[N_2^2 - (\omega - kU)^2]^{1/2}. \tag{4.9}$$

The radiating instability in this model exists for $U > U_c'$ and is observed in a narrow range of frequencies

$$kU - N_2 < \omega < kU$$

(see Fig. 13). We note that stratification in the bottom layer (water) can also lead to a radiating instability, which appears, however, on the bottom branch of the dispersion curve for much higher wind velocities $U > U_c$.[73]

4.3. Rossby waves on a tangential discontinuity

The presence of a sufficiently fast hydrodynamic flow that is capable of "overtaking" the waves incident on it still does not guarantee the possibility of super-reflection, for which a refracted wave with negative energy must be emitted into the moving medium. As an illustration we shall study the incidence of Rossby waves in the β plane on a TD.[82]

The perturbation of the stream function ψ in an incompressible rotating medium in the β plane satisfies the equation[27]

$$\frac{\partial}{\partial t}\Delta\psi + \beta\frac{\partial\psi}{\partial x} = 0. \tag{4.10}$$

This equation is suitable for disturbances whose characteristic frequencies are small compared with the rotational frequency of the medium $\Omega(y) = \Omega_0 + \beta y$ (the quasigeostrophic approximation).[3)]

The dispersion equation and the group velocity for Rossby waves follow from (4.10):

$$\omega = -\frac{\beta k}{k^2 + q^2}, \quad v_{\text{гр}} = \frac{\beta}{(k^2 + q^2)^2}(k^2 - q^2,\ 2kq). \tag{4.11}$$

For $\beta > 0$ only waves with negative phase velocity along the x axis are possible: $k < 0$. The y components of the phase and group velocities have the opposite sign:

$$q v_{\text{гр}\,y} = \frac{2\beta q^2}{(k^2 + q^2)^2} k < 0.$$

These relations determine the direction of the wave vectors in the incident, reflected, and refracted waves:

$$\psi(y<0) = \exp(iq_1 y) + R\exp(-iq_1 y),$$
$$\psi(y>0) = T\exp(iq_2 y), \tag{4.12}$$

where

$$q_1 = -[-\beta k\omega^{-1} - k^2]^{1/2} < 0,$$
$$q_2 = -[-\beta k(\omega - kU)^{-1} - k^2]^{1/2} < 0.$$

The boundary conditions on the TD permit joining the solutions (4.12) on the boundary $y = 0$. As a result we have[82]

$$R = \frac{q_1^2\omega^2 - q_2(\omega - kU)^2}{q_1\omega^2 + q_2(\omega - kU)^2}, \quad T = \frac{2q_1\omega(\omega - kU)}{q_1\omega^2 + q_2(\omega - kU)^2}. \tag{4.13}$$

It follows from the conditions $q_{1,2} < 0$ that the condition $|R| < 1$, always holds, i.e., the phenomenon of superreflection does not occur here. This is explained by the fact that the energy density in the refracted wave, propagating in a moving medium, is positive; indeed,

$$\mathscr{E} = \frac{\omega}{\omega - kU}\mathscr{E}_0 = \frac{\omega}{\omega - kU}(k^2 + q_2^2)|\psi|^2$$
$$= \frac{\omega(k^2 + q_2^2)}{-\beta k}|\psi|^2 > 0. \tag{4.14}$$

Figure 16 shows the different regimes of reflection for $\beta > 0$ as a function of the parameter $c = \omega/kU$. The range of normal reflection is separated from the range of total reflection by the angle

$$\tilde{\theta} = \arcsin\left(\frac{c}{c-1}\right)^{1/2}.$$

The condition $|R| = \infty$ gives a dispersion equation for characteristic waves on the TD in a rotating medium:

$$(\omega - kU)^2\left(k^2 + \frac{\beta k}{\omega - kU}\right)^{1/2} + \omega^2\left(k^2 + \frac{\beta k}{\omega}\right)^{1/2} = 0. \tag{4.15}$$

For the dimensionless parameter $C_0 = 2\omega/kU - 1$ it is not difficult to obtain from (4.15) the cubic equation $C_0^3 + 3SC_0^2 + C_0 + S = 0$ (where $S = \beta/2k^2U$), whose discriminant is $\Delta = -108S^4 + 36S^2 - 4 < 0$ and which therefore has two complex-conjugate roots. Thus the instability of

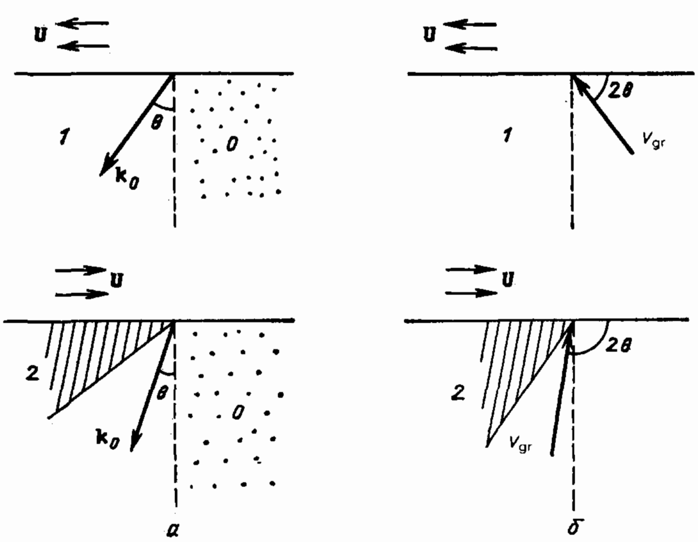

FIG. 16. Regimes of reflection of Rossby waves from TD. The ranges of the angles of incidence for the wave vector (a) and the group velocity (b). 0—Propagation impossible, 1—normal reflection, 2—total reflection

the TD remains also in the presence of rotation, while in the limit $S \to 0$ it transforms into the Kelvin-Helmholtz instability.

The questions of the stability of shear zonal flows in the β plane are studied in greater detail in Ref. 27.

5. RESONANCE INTERACTION OF WAVES WITH A FLOW

The "spreading" of a TD, i.e., replacement of the TD by a shear layer of finite width, not only significantly complicates the mathematical apparatus necessary for studying small oscillations, but it also leads to qualitatively new effects, which do not occur in piecewise-homogeneous flows of the TD type. The amplitude of the oscillations of the stream function ψ in a two-dimensional flow of an ideal incompressible liquid, whose velocity $\mathbf{V}_0 = (U(y), 0)$, satisfies Rayleigh's equation:

$$\frac{d^2\psi}{dy^2} - k^2\psi - \frac{d^2U/dy^2}{U - (\omega/k)}\psi = 0. \tag{5.1}$$

This equation contains a singularity (a critical layer) in a neighborhood of the resonance singular point y_c, where the velocity of the flow is equal to the phase velocity of wave disturbances: $U(y_c) = \omega/k$. In the critical layer the dynamics of small disturbances can no longer be described on the basis of Eq. (5.1)—more complicated models must be employed here. This was first done in Refs. 5 and 6 where a small viscosity, which removes the singularity in the equations of the linear theory, was introduced. In this case the Rayleigh equation transforms into the Orr-Sommerfeld equation:

$$\frac{i\nu}{k}\left(\frac{d^4\psi}{dy^4} - 2k^2\frac{d^2\psi}{dy^2} + k^4\psi\right) + \left(U - \frac{\omega}{k}\right)\left(\frac{d^2\psi}{dy^2} - k^2\psi\right) - \frac{d^2U}{dy^2}\psi = 0. \tag{5.2}$$

The investigation of the asymptotic behavior of the solutions of Eq. (5.2) in the limit $\nu \to 0$, which permits "joining" the solutions of Rayleigh's equation (5.1) in the neighborhood of the singular point y_c, is a problem in singular perturbation theory.[7]

A different, more physical, approach to the investigation of the dynamics of small disturbances in flows can be developed by comparing and generalizing analogous problems in systems of different physical nature. In particular, the plasma-hydrodynamic analogy, developed thus far, has turned out to be very heuristic for problems of hydrodynamic instability.[20] Here we shall study different aspects and possibilities of this approach.

5.1. The plasma-hydrodynamic analogy

Singularities arising in the equations for small disturbances at a resonance of the phase velocity of the disturbances with the velocity of the particles of the medium are typical for different waves in the medium. The appearance of singularities is best known for longitudinal waves in a plasma. The oscillations in a system of free electrons with the distribution function $f_i(v)$ under the action of a monochromatic electrostatic field $E_0\exp(-i\omega t + ikx)$ are described by a linearized kinetic equation:

$$-i(\omega - kv)f + \frac{eE}{m}\frac{\partial f_0}{\partial v} = 0. \tag{5.3}$$

Calculating the current

$$j = e\int fv\,dv,$$

we obtain the following expression for the conductivity of the plasma:

$$\sigma = \frac{e^2}{m}\int\frac{\partial f_0/\partial v}{i(\omega - kv)}\,dv, \tag{5.4}$$

where the integrand has a singularity at $v = \omega/k$.

The rule for integrating around this singularity (Landau's rule) is obtained by solving the initial-value problem by Laplace's method.[84] In the process, the growing ($\mathrm{IM}\,\omega > 0$) field is studied, and in making the analytic continuation to real values of ω the contour of integration in (5.4) must be deformed so as to go around the singularities in the integrand all the while remaining below them. This gives Landau's rule for integrating around the singularities for real values of ω and k: the contour of integration in (5.4) goes around the singularity $\omega = kv$ from below. As a result the quantity $\mathrm{Re}\,\sigma$, characterizing the damping of the field, is determined by the half-residue of the integrand:

$$\mathrm{Re}\,\sigma = -\frac{\pi e^2}{m}\left(\frac{\partial f_0}{\partial v}\right)_{v=\omega/k}. \tag{5.5}$$

Thus the sign of $\mathrm{Re}\,\sigma$ is determined by the derivative of the velocity distribution function with the particle velocity equal to the phase velocity of the wave. This is actually connected with the fact that the amount of work performed by a weak wave on the particles is determined by which type of particles are more numerous when $v = \omega/k$: particles that overtake the wave or particles that lag behind it. If more particles overtake the wave ($\partial f_0/\partial v > 0$), then $\mathrm{Re}\,\sigma < 0$ and the wave grows.

The analogy between hydrodynamic and electrodynamic phenomena in flows of particles in the perspective studied here originated in the work of Case,[9] it was then discussed by Timofeev,[10] and it was analyzed in detail in Ref. 20. This analysis made it clear that the relationship between electrodynamic and hydrodynamic phenomena is more profound than noted previously. We shall see below, however, that there are singularities that are specific to hydrodynamics and which do not have any electrodynamic analogs.

In a hydrodynamic flow particles with different velocity are located at different levels y, and for this reason singular resonance points arise in differential equations of the type (5.1). From the viewpoint of the analytical theory of differential elquations[85] these singularities are logarithmic branch points, if y_c is not a point of inflection of the velocity profile, i.e., $d^2U(y_c)/d^2y \neq 0$. The question of the relation between the analytical solutions on both sides of y_c was solved by Lin[6] and Wasow.[7] Their method is based on introducing a small viscosity and constructing an asymptotic expansion of the solutions of the Orr-Sommerfeld equation (5.2) as a function of ν. The result can be summarized as follows.

The solution of Rayleigh's equation is a limiting case of the viscous solution obtained from the Orr-Sommerfeld equation in the limit $\nu \to 0$, if the branches of the multivalued nonviscous solution near the branch points y_c are chosen using Lin's rule for integrating around the singularities: the contour of integration passes in the complex y plane below the points y_c if $\mathrm{d}U(y_c)/\mathrm{d}y>0$ and above them if $\mathrm{d}U(y_c)/\mathrm{d}y<0$.

It should be noted that the inclusion of a small dissipation (particle collisions) in a plasma also removes the resonance singularity for Langmuir waves in the expression for the conductivity σ and makes it possible to derive Landau's rule for integrating around the singularities. Viscous dissipation in hydrodynamics, however, corresponds to the highest order derivative in the Orr-Sommerfeld equation. The construction of an asymptotic solution of the corresponding problem with singular disturbances—the solution of the differential equation with a small parameter in front of the highest order derivative—is a relatively difficult problem.

At the same time Lin's rule can also be derived by the traditional "plasma" method: the initial-value problem can be solved with the help of a Laplace transformation in time.[9,86] In so doing complex values of ω in the half-plane $\mathrm{Im}\,\omega>0$, which corresponds to growing solutions, are studied. Lin's rule arises in a natural manner with the analytic continuation of the solution to real values of ω.

Thus Laplace's method reveals a profound analogy with the plasma problem solved by Landau. This analogy is manifested not only in the final result—the rule for integrating around singular points—but also in different effects associated, in particular, with the behavior of the solutions from the continuous spectrum (see Refs. 9 and 87).

The existence of resonance points, together with a corresponding rule for integrating around them, determines the mechanism responsible for the exchange of energy between a weak disturbance and the average flow. We note that the resonance mechanism of damping (or amplification) of waves is not directly related with viscous dissipation, so that resonance interaction appears in an ideal liquid. In what follows we shall study this interaction in simple examples, where the similarity and difference between hydrodynamic problems and the corresponding problems in plasma physics will be evident.

5.2. Resonance amplification of sound

The study of the reflection of sound from a flow with a continuous velocity profile[88–90] is a much more difficult problem than Miles' problem of reflection from a TD. We shall employ the expressions (3.1), which are the asymptotic solutions in the limits $y\to\pm\infty$, where the velocity of the flow is $U(y)\to$const. The behavior of these solutions in the region of the shear ($|y|\lesssim l$) is determined by the equation for the amplitude of the pressure[1]:

$$\frac{\mathrm{d}^2P}{\mathrm{d}y^2}-\frac{2\,\mathrm{d}U/\mathrm{d}y}{U-(\omega/k)}\frac{\mathrm{d}P}{\mathrm{d}y}+\left[\frac{(U-\omega k^{-1})^2}{c^2}-k^2\right]P=0. \quad (5.6)$$

For supersonic flows this equation, like Rayleigh's equation (5.1) in an incompressible liquid, contains a singular point y_c, where $U(y_c)=\omega/k$, and the solution $P(y)$ has, in general, a branch point. The choice of the branch of a multivalued solution is determined by Lin's rule.

The resonance interaction of a sound wave, occurring in the critical layer in the limit $y\to y_c$, can lead to a change in the energy of the sound. The criterion showing the direction of exchange of energy between the wave and the flow can be obtained from (5.6) in a general form. Multiplying (5.6) by $p^*\times(U-c)^{-2}$ (where P^* is the complex-conjugate function) and integrating by parts taking into account the boundary conditions (3.1) and using Lin's rule gives the law of conservation of energy in the form

$$\frac{q_1}{c^2}(1-|R|^2)-\frac{q_2}{(U-c)^2}|T|^2=\pi k^2\left[|p|^2\frac{\mathrm{d}^2U\,|\,\mathrm{d}y^2}{|\,\mathrm{d}U/\mathrm{d}y\,|^3}\right]_{y=y_c}. \quad (5.7)$$

The left side of this equality characterizes the relative absorption of sound and represents the difference between the energy flux of the incident wave (with unit amplitude) and the sum of the energy fluxes for the reflected and refracted waves. The sign of the right side of (5.7), characterizing the absorption of energy in the critical layer, is determined by the slope of the velocity profile at the resonance point. The concave sections of the profile $\mathrm{d}^2U/\mathrm{d}y^2<0$ amplify the sound wave, while the convex sections $\mathrm{d}^2U/\mathrm{d}y^2>0$) absorb the sound wave.

The relation between the increment and the slope of the profile U has a simple physical intepretation.[90] Namely, the sign of the second derivative of the velocity profile at the resonance level determines the ratio of the number of resonant particles overtaking the wave and the number of particles lagging behind the wave. If in the neighborhood of a resonance level there are more particles overtaking the wave than lagging behind it, then the wave is amplified, extracting energy from the flow in this neighborhood. Indeed, we shall study the velocity distribution function of the particles in the flow—the number of particles per unit area and per unit velocity interval:

$$f(U)=\frac{\mathrm{d}y}{\mathrm{d}U}=\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)^{-1}. \quad (5.8)$$

The ratio of the number of particles overtaking and lagging behind the wave is determfined by the derivative $\mathrm{d}f/\mathrm{d}U=-\mathrm{d}^2U/\mathrm{d}y^2[\mathrm{d}U/\mathrm{d}y]^{-3}$—in complete agreement with the criterion (5.7). This relation between the amplification (damping) of the wave and the derivative of the velocity distribution function of the particles is completely analogous to the condition of amplification of plasma waves studied above. This analogy is obvious, since both mechanisms are determined by the resonant particles, moving with a velocity equal to the base velocity of the wave.

Rayleigh's well-known theorem,[4] relating the instability of shear flows with the presence of a point of inflection in the velocity profile, can be interpreted analogously from the viewpoint of the plasma-hydrodynamic analogy. For this we shall employ the necessary condition for the kinetic instability of Langmuir waves: the velocity distribution function of the particles must have a minimum.[14] For a shear flow the extrema of the function $f(U)$, at which $\mathrm{d}f/\mathrm{d}U=0$, are located precisely at the points of inflection, where $\mathrm{d}^2U/\mathrm{d}y^2=0$. Thus Rayleigh's theorem is the hydrodynamic variant of the "plasma" criterion.[14]

The coefficients of reflection and refraction of sound for an arbitrary continuous profile can be calculated approximately, if the boundary-value problem (5.6) and (3.1) contains a small parameter. For a "narrow" shear layer ($\mu\equiv kl\ll 1$) corrections $\sim\mu$ to the expressions (3.2) for the TD can be obtained by the method of joined asymptotic expansions. In the region of total reflection, when there is no refracted wave, the relative amplification of the reflected wave is small[4),90] ($1-|R|^2 \sim \mu$, and is determined exclusively by the resonant interaction with synchronous particles—the mechanism of superreflection, associated with the presence of a refracted wave with negative energy, does not operate here.

The other limiting case of smoothly inhomogeneous flows ($\mu \gg 1$) also admits an approximate solution.[89] The resonance interaction is exponentially weak here, since the critical level y_c lies in the region of non-transmission, where the field of the wave decays exponentially.

The position of the critical layer for a given velocity profile is determined for sound waves by the angle of incidence:

$$\sin\theta = \frac{ck}{\omega} = \frac{c}{U(y_c)}.$$

This position is also uniquely fixed by the sign of the resonance interaction

$$(1-|R|^2) \sim \left.\frac{d^2U}{dy^2}\right|_{y=y_c}.$$

The magnitude of this effect reaches a maximum for $\mu \sim 1$[88] and depends on the shape of the velocity profile.

5.3. Resonance mechanism for generation of wind waves

An interesting and important example illustrating the resonance mechanism of amplification of wind-generated disturbances in a shear flow is the well-known mechanism of Miles for generation of wind waves.[91] Thus far this mechanism has been studied in greatest detail on the basis of a model of a quasilaminar air flow above the surface of deep water. In this model the motion of the air is assumed to be plane-parallel, and in studying its small fluctuations the viscosity and nonlinear effects are neglected, i.e., Rayleigh's equation (5.1) for the amplitude of the velocity fluctuations is employed. At the same time the velocity profile $U(y)$ of the undisturbed flow is chosen as the profile that is realized for the average velocity of the turbulent boundary layer above a smooth solid surface—the so-called logarithmic boundary layer.[25] Thus the presence of turbulent pulsations in the wind (unrelated with surface waves) is taken into account only in the choice of the velocity profile of the shear flow and is ignored when small fluctuations of this flow are studied.

The use of such approximations leads to a simple model of a shear flow of an ideal liquid above the interface—the "carrier" of surface gravity waves—between two media with different density. To find the growth increment of the surface waves we shall integrate, following Miles,[91] Rayleigh's equation (5.1), multiplying it first by ψ^* and applying the boundary conditions on the surface of the water. This gives the following expression for the growth increment of surface waves[91,20]:

$$\gamma = -\frac{\rho_a}{\rho_W}\frac{\pi\omega^3}{2k}\left(\frac{d^2U/dy^2}{|dU/dy|^3}\right)_{y=y_c}\left|\frac{P(y_c)}{P(0)}\right|^2. \tag{5.9}$$

The dependence of the amplification of the wind waves on the slope of the velocity profile here is the same as the dependence (5.7) of the amplification of sound waves in a shear flow. By its very nature the Maxwellian mechanism for generation of wind waves is analogous to the mechanism of the kinetic instability of waves in a plasma; the sign of the increment (5.9) is defined, as for sound, by the derivative of the velocity distribution function of the particles in the shear flow:

$$\left.\frac{df}{dU}\right|_{y=y_c} \sim \left.\frac{dU}{dy}\right|_{y=y_c}.$$

In the boundary layer above the water surface, where $d^2U/dy^2 < 0$, the resonance interaction with the wind-generated flow amplifies surface waves. A calculation of Miles' increment for a logarithmic profile gives, in many cases, values that are close to the experimentally measured values.[25]

Resonance amplification of surface waves in the critical layer of a shear flow occurs not only for wind waves. Miles' theory has been employed to study the generation of acoustic waves on a delay structure,[20] magnetohydrodynamic waves in the clouds of radio galaxies,[92] etc.

5.4. Internal gravity waves in a critical layer

Resonance interaction of wave disturbances with a stratified flow has a number of characteristic distinguishing features. The Taylor-Goldstein equation[26]

$$\frac{d^2\psi}{dy^2} - k^2\psi - \frac{d^2U/dy^2}{U-(\omega/k)}\psi + \frac{N^2}{[U-(\omega/k)]^2}\psi = 0, \tag{5.10}$$

which extends Rayleigh's equation (5.1) to stratified media ($N \neq 0$), has at the critical point y_c a singularity of a higher order than (5.1).[5)] Here the Richardson number $\mathrm{i} = N^2(dU/dy)^{-2}$ evaluated at the critical point plays the fundamental role. For $\mathrm{Ri}_c \equiv \mathrm{Ri}(y_c) > 1/4$ the energy of IGWs is absorbed in the critical layer.[26] The interpretation of this result is based on the use of the WKB method for $\mathrm{Ri} \gg 1/4$ in the neighborhood of the critical layer, where a wave packet with decreasing wavelength approaches the critical point over an infinitely long time.

At the same time for $\mathrm{Ri}_c < 1/4$ the resonance interaction of IGW with the flow in the critical layer can lead to amplification of the wave. Some of the articles 95–98 in which this possibility was examined are based on the use of model (piecewise-linear, etc.) velocity profiles, the application of numerical methods, and the selection of profiles $U(y)$ and $N(y)$ for which Eq. (5.10) can be solved exactly with the boundary conditions

$$\begin{aligned}\psi(-\infty) &= \exp(iq_1z) + R\exp(-iq_1z),\\ \psi(+\infty) &= T\exp(iq_2z),\end{aligned} \tag{5.11}$$

where

$$q_1 = -k[N^2(-\infty)-\omega^2]^{1/2}/\omega,$$
$$q_2 = -[N^2(+\infty)-(\omega-kU(+\infty))^2]^{1/2}[\omega-kU(+\infty)]^{-1}.$$

A criterion for resonance amplification for "narrow" shear flows $U = U_{0\varphi}(y/l)$ ($\mu = k_0 l \ll 1$), for which the characteristic Richardson's number $\mathrm{Ri} \sim N^2l^2/U_0^2 \ll 1$, was found in Ref. 77. The method of joined asymptotic expansions in the parameter μ enables finding the coefficients R and T for an arbitrary "narrow" shear layer, combined with an arbitrary "narrow" density differential, determining the local maximum of the Brunt-Väisälä frequency in the region of the shear: $N^2(y) = N_0^2 + N_1^2(y/l)$, $N_1^2/N_0^2 \sim \mu$. The relative amplification $\Pi = |R|^2 + (\mathrm{Re}\, q_2/q_1)|T|^2 - 1$, characterizing the excess of the sum of the energy fluxes of the reflected and refracted waves above the unit energy flux of the incident wave and found by this method, has the form

$$\Pi = \frac{4\pi\,|q_1 q_2^2|\,(\omega/k)^2\,[U_0-\omega/k)]^4}{\{q_1(\omega/k)^2+q_2[U_0-\omega/k)^2]\}^2+\left(\int\limits_{-\infty}^{+\infty} N_1^2\mathrm{d}y\right)^2} \times\left\{\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{|P(y)/P(\infty)|^2}{\mathrm{d}U/\mathrm{d}y}\right]\right\}_{y=y_c}. \quad (5.12)$$

The asymptotic expansion of the amplitude of the pressure to a first approximation in μ

$$P(y) = P(\infty)\left[1 - iq_2^{-1}\left(U_0-\frac{\omega}{k}\right)^{-2}\int\limits_y^\infty N_1^2\mathrm{d}y\right] = P(\infty)+\rho_0\eta\int\limits_y^\infty N_1^2\mathrm{d}y, \quad (5.13)$$

where η is the amplitude of the displacement of the isopycnic surface, appears here.

It is obvious from (5.12) that IGWs grow ($\Pi > 0$) when they are reflected from the shear flow, if

$$\left[\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}y}\frac{|P|^2}{\mathrm{d}U/\mathrm{d}y}\right]_{y=y_c} > 0. \quad (5.14)$$

This criterion has a simple physical meaning, based on the fact that the quantity

$$f = \left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)^{-1}$$

is the velocity distribution function of the particles in the flow. In an unstratified flow

$$\left.\frac{\mathrm{d}|P|^2}{\mathrm{d}y}\right|_{y=y_c} = 0;$$

in this case the gain is proportional to the derivative

$$\frac{\mathrm{d}f}{\mathrm{d}y} = -\frac{\mathrm{d}^2U}{\mathrm{d}y^2}\left(\frac{\mathrm{d}U}{\mathrm{d}y}\right)^{-3},$$

evaluated at the critical point y_c.

In the presence of significant stratification ($N_1 \neq 0$, $\mathrm{Ri}_c \gtrsim \mu$) the intensity of the wave field (to a first approximation in μ) becomes nonuniform:

$$\left.\frac{\mathrm{d}|P|^2}{\mathrm{d}y}\right|_{y=y_c} \neq 0.$$

According to (5.13) this nonuniformity is determined by the gradient of the amplitude of the hydrostatic pressure fluctuations which arise with the periodic displacement of a layer of liquid with variable density. Because the field of the wave is nonuniform in the critical layer the efficiency of the interaction of the field with the particles that overtake and lag behind the wave is different: the direction of energy exchange is now determined, according to the criterion (5.14), by the derivative of the quantity $|P|^2 f(U)$.

Attributing the resonance interaction to the competition between absorption and amplification as the overtaking and lagging particles interact with the disturbance also makes it possible to explain the absorption of IGWs in a critical layer with a large Richardson's number. In this case the wave is absorbed by the lagging particles in the flow, while only an insignificant part of the wave field reaches the overtaking particles, capable of amplifying the wave.

Because the amplification (absorption) of the wave depends on the character of the wave field in the critical layer the criterion of amplification does not reduce to a local condition of the Miles' criterion type $\mathrm{d}^2U(y_c)/\mathrm{d}y^2 < 0$. The amplification of IGWs in a stratified flow is determined by a global condition, which includes the characteristics of the flow in the entire region studied and not only in the critical layer. A more general condition than (5.14) for amplification, which is valid for arbitrary Richardson's numbers, can be derived. For this we shall employ the equation for the amplitude of the pressure:

$$\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}P/\mathrm{d}y}{[U-(\omega/k)]^2-(N^2/k^2)} - \frac{k^2}{[U-(\omega/k)]^2}P = 0. \quad (5.15)$$

We multiply (5.15) by P^* and integrate by parts. The imaginary part of the equality obtained gives the criterion for amplification in the form[77]:

$$\Pi = -\frac{1}{\rho_0|q_1|\,|P(-\infty)|^2}\mathrm{Im}\int\limits_{-\infty}^{+\infty}\frac{|P|^2\mathrm{d}y}{[U-(\omega/k)]^2} > 0. \quad (5.16)$$

The criterion (5.14) follows from (5.16), if the asymptotic expression to a first approximation in μ, not containing any singularity at the point y_c, is employed for the pressure in the integrand and a transformation is made into the upper half-plane of the complex frequency $\omega = \omega_r + i\delta$, having in mind the solution of the initial-value problem by Laplace's method. We emphasize that the exact expression for the pressure amplitude has a singularity at $y = y_c$. This does not permit displacing the integration path in (5.16) into the complex y-plane, i.e., reducing the integration in (5.16) as $\delta \to 0$ simply to Lin's rule.

The results of the numerical calculation of resonance growth of IGWs for a shear layer with the profile $U = (U_0/2)[1 + \mathrm{th}(y/l)]$, $N = \mathrm{const}$ are presented in Fig. 17. The figure also shows contour lines of the maximum growth $\Pi_{\max} = \max\,\Pi(\theta)$ in the plane of the parameters ω/kU_0, $\mathrm{Ri} = 4N^2l^2/U_0^2$. We note that resonance growth ($\Pi_{\max} > 0$) is possible not only for $\omega/kU_0 > 0{,}5$, when the critical point lies on the convex section of the velocity profile, as happens for waves in an unstratified liquid, but also for $\omega/kU_0 < 0{,}5$, when the waves are in resonance with the concave section of the profile.

An important feature of resonance growth of IGWs is the possibility of spontaneous emission of such waves: for some values of the parameters Ri, ω/k, 0 the quantity $\Pi \to \infty$ (the contour line $\Pi_{\max} = \infty$ in Fig. 17). This corresponds to resonance between the incident wave and the neutral characteristic mode of the system, in which above the shear layer ($y \to \infty$) the field decays exponentially while below this layer ($y \to -\infty$) the field is a traveling wave. Such neutral disturbances, corresponding to spontaneous emission of IGWs by the shear layer, were found in Refs. 99 and 100. We emphasize that unlike spontaneous emission of TD here the energy of the IGWs is not conserved: there is a critical layer where the wave extracts energy from the stratified flow.

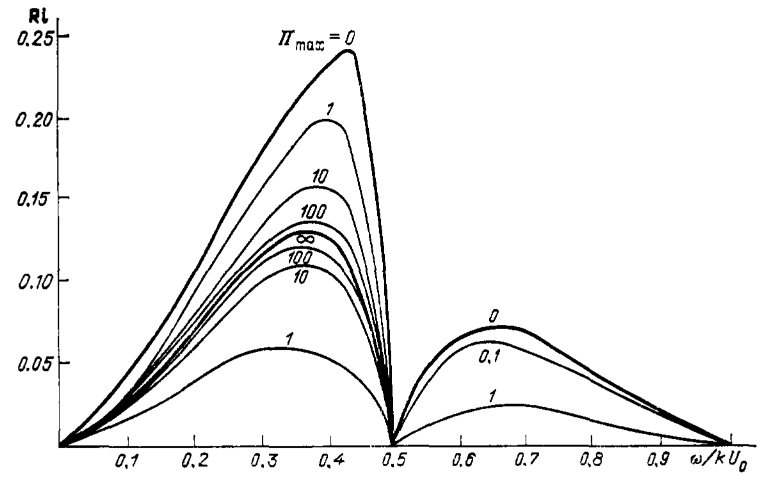

FIG. 17. Resonance amplification of IGWs on a shear layer: the isolines Π_{max} = const in the parameter plane Ri, ω/kU_0.

6. SCATTERING OF WAVES BY VORTICES

6.1. Algebraic method for cylindrical vortices

The properties of waves in shear flows, studied above, can also be observed in flows with closed streamlines—cylindrical vortices. Small disturbances in axisymmetric flows are usually studied by an algebraic method, based, as in the case of plane-parallel flows, on approximation of the velocity distribution with a profile with piecewise-constant vorticity and joining analytical solutions on the boundaries of the regions.

In polar coordinates the amplitude of disturbances of the form $\exp(-i\omega t + in\varphi)$ for the stream function satisfies an equation analogous to (5.2):

$$\hat{L}\psi + \frac{n\, dQ/dr}{r(\omega - n\Omega)}\psi = \frac{i\nu n}{\omega - n\Omega}\hat{L}^2\psi, \qquad (6.1)$$

where $\hat{L} = d^2/dr^2 + r^{-1}d/dr - (n^2/r^2)$. For $\nu = 0$ Eq. (6.1) is analogous to Rayleigh's equation; it contains the singular points $\omega - n\Omega = 0$ with a coefficient proportional to the derivative of the vorticity $dQ/dr = r d^2\Omega/dr^2 + 3d\Omega/dr$. In Rayleigh's equation the residue of the coefficient at the singular point had the same meaning and was proportional to the second derivative of the velocity profile. It is also possible to prove an assertion analogous to Rayleigh's theorem[101]: disturbances growing in time can exist only if there exist points of an extremum of the vorticity where $dQ/dr = 0$ in the profile of the angular velocity (in a plane-parallel flow these are points of inflection of the velocity profile).

For vortices with constant vorticity, in which $\Omega = \Omega_0 + (\varkappa/r)$, Eq. (6.1) has exact solutions of the form $\psi = \psi_a = Ar^n + Br^{-n}$. This makes it possible to employ the algebraic method, using profiles of the angular velocity consisting of several parts with uniform vorticity and delimited by tangential discontinuities or breaks in the rate of change of the velocity. For such a "piecewise" profile it is not difficult to find a solution by joining expression of the type ψ_a in each part with the help of boundary conditions on the discontinuities.[23] In particular, for a cylindrical TD (Fig. 18a) we have the dispersion equation $\omega^2 + (\omega - n\Omega_0)^2 = |n|\Omega_0^2$, which transforms into the Kelvin–Helmholtz equation in the limit $n \to \infty$ (in this case $n\Omega_0 \to kU$). Its solution

$$\omega = \frac{1}{2}\Omega_0[n \pm (2|n| - n^2)^{1/2}]$$

corresponds to unstable modes for all $n \geqslant 2$. The mode $n = 1$, corresponding to displacement of the vortex as a whole, is obviously neutrally stable.

For a different model—Rankine's vortex (Fig. 18b) the dispersion equation describing neutral stable oscillations has the form $(\omega - n\Omega_0)[\omega - (n-1)\Omega_0] = 0$.

Within the framework of the algebraic method it is also possible to determine, with the help of a model, the effect of viscosity on oscillations of vortices. For this it is sufficient to study a discontinuity between a viscous liquid and an ideal liquid, i.e., to take into account the viscosity, for example, only in the nucleus of the vortex. Taking into account viscosity with $dQ/dr = 0$ does not change the solutions ψ_a since they identically satisfy Eq. (6.1). Increasing the order of the equation, however, with $\nu \neq 0$ leads to the appearance of additional linearly independent solutions, which for large Reynold's numbers Re oscillate strongly and decay along the coordinate r as the distance from the source increases. The boundary conditions on the discontinuity at $r = a$—continuity of the components of the momentum of the flow which are normal to the mobile boundary—permit joining

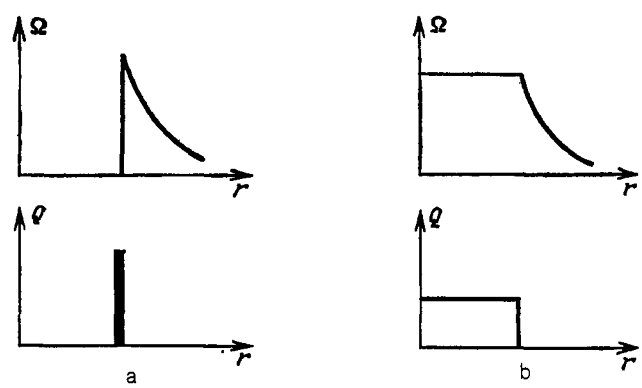

FIG. 18. Profiles of the angular velocity and vorticity for different cylindrical vortices. a—cylindrical TD. b—Rankine's vortex.

nonviscous solutions in the region outside the vortex and a linear combination of nonviscous and damped viscous oscillations in the nucleus of the vortex. This method gives for the oscillations of the Rankine vortex, to a first approximation in the parameter $\mathrm{Re} = \nu/\Omega_0 a^2$, the dispersion equation

$$\omega = (n-1)\,\Omega_0 - \frac{2i\nu}{a^2}\, n(n-1), \tag{6.2}$$

which corresponds to damped oscillations. The interpretation of this result must take into account the fact that in a medium rotating more rapidly than the angular phase velocity of azimuthal waves ($\omega - n\Omega_0 < 0$), dissipation is negative. However the energy of the oscillations of the vortex is also negative[102]

$$E = -\pi a^4 \Omega^2 \frac{n-1}{n} \rho \left| \frac{\xi_0}{a} \right|^2 < 0, \tag{6.3}$$

where ξ_0 is the amplitude of the displacments of the boundary of the vortex nucleus.

The negative energy of the characteristic oscillations of the vortex makes possible radiating instability of the vortex in a compressible medium.[103] For $\mu \equiv \Omega_0/c < 1$ the oscillations are close to incompressible oscillations for $r \leqslant a$, but the non-steady-state flow associated with them emits sound in the wave zone $r \gtrsim \lambda$. As a result of the extraction of energy by the outgoing sound waves the amplitude of the oscillations grows.

6.2. Amplification of sound by vortices

The existence of negative-energy waves in a rotating medium makes possible, in principle, amplification of a sound wave incident on the vortex. This, however, requires a negative–energy sink; in a plane–parallel TD the escape of refracted waves to infinity played the role of just such a sink. For vortices in which the region of the nucleus is finite this is not possible. Here, however, amplification of scattered sound is possible owing to a change in the sign of viscous dissipation in the vortex nucleus.

Consider the scattering of sound by a vortex of size $a \ll \lambda = 2\pi c/\omega$.[23,24] The field in the far zone

$$\begin{aligned} P &= \exp\left(i\,\frac{\omega r \cos\theta}{c}\right) + f(\theta)\, r^{-1/2} \exp\left(i\,\frac{\omega r}{c}\right) \\ &= \left(\frac{2\pi\omega r}{c}\right)^{-1/2} \sum_{n=-\infty}^{\infty} i^n e^{in\varphi} \\ &\quad \times \left\{ \exp\left[-i\left(\frac{\omega r}{c} - \frac{\pi n}{2} - \frac{\pi}{4}\right)\right] \right. \\ &\quad \left. + R_n \exp\left[i\left(\frac{\omega r}{c} - \frac{\pi n}{2} - \frac{\pi}{4}\right)\right] \right\} \end{aligned} \tag{6.4}$$

is determined by the coefficients of reflection of cylindrical harmonics

$$R_n = 1 + \left(\frac{2\pi i\omega}{c}\right)^{1/2} f_n,$$

where

$$f(\theta) = \sum_n f_n \exp(in\theta)$$

is the scattering amplitude. Energy exchange between the wave and the vortex is characterized by the quantities $|R_n|$, which can be found to a first approximation in the parameter $\mu \equiv a/\lambda \ll 1$. For this it is sufficient to join the solution for $r \ll \lambda$, obtained by the algebraic method, and the solution for $r \gg a$ in the form of a sum of incident and reflected cylindrical waves. For Rankine's vortex with a viscous nucleus it is not difficult to obtain by this method[23]

$$1 - |R_n|^2 = \frac{8\pi\nu/a^2}{(|n|-1)!\,(|n|-2)!} \left(\frac{\omega a}{2c}\right)^{2|n|} \frac{\omega - n\Omega_0}{[\omega - (n-1)\,\Omega_0]^2}. \tag{6.5}$$

For $\omega - n\Omega_0 < 0$ sound is amplified: $|R_n| > 1$. The amplification of sound by a rotating viscous vortex represents the acoustic analog of the effect studied in Ref. 104, where it was shown that electromagnetic waves can be amplified when they are scattered by a rotating, conducting cylinder and gravitational waves can be amplified when they are scattered by a collapsing rotating body.

It should be noted, however, that the mechanism of viscous dissipation of sound in a vortex flow does not reduce to simple absorption, but rather is determined by linear transformation into rapidly decaying vortex waves.

In vortex flows with a smooth profile and $dQ/dr \neq 0$ the resonance mechanism of absorption (amplification) of weak disturbances, analogous to the mechanism of cyclotron absorption of waves in magnetoactive plasma,[23] is also possible. This mechanism exists for $\nu = 0$, when Eq. (6.1) contains singularities where $\omega - n\Omega = 0$. The integration around these singularities is performed in the same manner as in the case of rectilinear shear flows following Lin's rule.[6]

Following Refs. 23 and 24, we shall examine the resonance mechanism of amplification of sound scattered by a cylindrical vortex with an arbitrary, continuous profile of the angular velocity $\Omega(r)$. For this we shall calculate the quantity $L(r) = \mathrm{Re}(r^2 u^* v)$, where $\mathbf{V} = (u, v)$ is the amplitude of the velocity fluctuations. For $r \gg a$ (a is the size of the vortex nucleus) the flow is a potential flow; it is not difficult to derive for the potential Φ the equation[24]

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} - \frac{n^2}{r^2}\Phi + \frac{(\omega - n\Omega)^2}{c^2}\Phi = 0. \tag{6.6}$$

Multiplying (6.6) by $inr\Phi^*$ and integrating by parts from $r = r_1 (a \ll r_1 \ll \lambda)$ to $r = r_0 \gg \lambda$ we find that $L(r_1) = L(r_0)$. For $\mu = \omega a/r \ll 1$ and $\Omega \lesssim \omega$ the approximate solution of Eq. (6.6)

$$\Phi = A\left[H_n^{(1)}\left(\frac{\omega r}{c}\right) + R_n H_n^{(2)}\left(\frac{\omega r}{c}\right)\right], \tag{6.7}$$

where $H_n^{(1)}$ and $H_n^{(2)}$ are Hankel functions of the first and second kind, gives

$$L(r_0) = \frac{2|n|}{\pi} |A|^2 (|R_n|^2 - 1). \tag{6.8}$$

At the same time, for $r \ll \lambda$ the medium can be regarded as incompressible and Eq. (6.1) (with $\nu = 0$) can be employed. Multiplying this equation again by $inr\psi^*$ and integrating by parts, applying the rule for integrating around the singular points, we obtain

$$L(r_1) = -\pi n \left(|\psi|^2 \left| \frac{d\Omega}{dr} \right|^{-1} \frac{dQ}{dr} \right)_{r=r_n},$$

where $\Omega(r_n) = \omega/n$. Comparing this expression with (6.8) we note that

$$\psi = B\left[\left(\frac{r}{a}\right)^n + q_n\left(\frac{r}{a}\right)^{-n}\right].$$

Thus if the vorticity decays as r increases, then the sound wave is amplified on reflection from the vortex ($|R_n|>1$). The vorticity usually decays monotonically away from the axis of the vortex, and the sound wave is therefore amplified by the vortex.

For $r \gg a$ the solution of Eq. (6.1) has the form

$$1-|R_n| \sim \left.\frac{dQ}{dr}\right|_{r=r_n}. \qquad (6.9)$$

It follows from here that $L(r_1) = -2n^2|B|^2 \mathrm{Im} q_n$. Then, using the formula (6.8), we obtain

$$1-|R_n|^2 = \frac{4\pi}{|n|!(|n|-1)!}\left(\frac{\omega a}{2c}\right)^{2|n|} \mathrm{Im}\, q_n. \qquad (6.10)$$

The dimensionless quantities q_n are determined by the profile of the relative angular velocity Ω/ω. For $n = \pm 1$ the solution of Eq. (6.1) has the form

$$\psi = Ar(\Omega \pm \omega),$$

and therefore the quantities $q_{\pm 1}$ are real. Thus in an ideal liquid reflection of the first cylindrical harmonics is elastic. The resonance mechanism of dissipation, associated with the existence of the singular points r_n, operates only for $n \geqslant 2$. In particular, Fig. 19 shows the result of the numerical calculation of Im q_2[23] for the profile

$$\Omega(r) = \Omega_0\left(\frac{r}{a}\right)^{-2}\left[1-\exp\left(-\frac{r^2}{a^2}\right)\right].$$

The cross section for absorption of a plane wave by a vortex is determined by the harmonic with $n = 2$:

$$\sigma_r = \frac{c}{\omega}\sum_{n=-\infty}^{\infty}(1-|R_n|^2) \approx \frac{c}{\omega}(1-|R_n|^2)$$
$$= \frac{\pi a}{8}\left(\frac{\omega a}{c}\right)^3 \mathrm{Im}\, q_2.$$

For vortices with decreasing vorticity ($dQ/dr<0$) the absorption cross section is negative; in this case the scattering of a wave by the vortex is accompanied by amplification of the wave. We also note that for $\Omega \sim \omega$ the absorption cross section is of the same order of magnitude $\mu^3 a$ as the scattering cross section.[23,24]

7. CONCLUSIONS

This review covers a realtively small range of phenomena and problems associated with the propagation of waves in shear hydrodynamic flows. Because of the limited space available here many important and interesting questions directly related with the subject of the review as well as pertaining to closely allied areas were omitted. We shall list some of these questions, which in our opinion should at least be mentioned briefly.

Interaction with shear flows has been studied many times as a mechanism for generation and amplification of different types of waves in a plasma.[14] The characteristic oscillations and stability of TD in magnetohydrodynamics were studied in Ref. 105, superreflection of magnetohydrodynamic waves from TD was studied in Refs. 106–108, and superreflection of electromagnetic waves from TD was studied in Ref. 109. For resonance interaction of electromagnetic waves with shear flows in a magnetic field it was found that the exchange of energy between the wave and the flow in the critical layer depends on the derivative of the velocity distribution function of the particles in the hydrodynamic flow.[110] The interesting possibility of a transition between plasma and hydrodynamic criteria for resonance amplificaion was established in Ref. 111, which is devoted to the study of waves in the critical layer of a plasma flow in the kinetic approximation. In this case the phase velocity of the wave is synchronized not only with the particles at the critical level, which move with the average velocity of the flow, but also with particles in other layers whose velocity deviates from the average value.

Nonlinear effects in the interaction of waves with a flow form another group of problems that were omitted from this review. Here the plasma-hydrodynamic analogy is also employed extensively.[20] The nonlinear dynamics of a critical layer, analogous to the dynamics of plasma particles in resonance with a Langmuir wave in velocity space, has been studied in Refs. 112–115. A quasilinear theory[21] and a theory of stimulated Raman scattering by particles of the flow[22] for wind waves in the ocean have been developed. The hydrodynamic analog of the plasma echo in a shear flow was studied in Ref. 116.

The analogy to problems in plasma physics and electronics makes possible a new approach to the theory of aerodynamic sound generators,[28,117] in particular, it makes possible new types of sound generators that are similar to existing electronic devices.[20]

In this review the theory of oscillations of vortices and their interaction with external wave fields has been studied very briefly. A separate review could, in principle, be devoted to this subject. Here, however, in our opinion, there are many unsolved or inadequately studied problems. They include, in particular, some problems in geophysical hydrodynamics: emission of atomspheric waves by cyclones and tornadoes,[118,119] interaction of different types of waves with oceanic vorticies,[120] etc.

It would be natural to attempt to extend the results presented here to different types of waves in the atmosphere and the ocean (Kelvin and Poincare waves, inertial gyroscopic waves, trapped shelf waves[27]), in the earth's magnetosphere,[121] in the solar wind,[122] and in the sun's atmosphere.[123] Also of interest is a study from this perspective of waves in realtivistic flows (we call attention to Refs. 124–126 which are devoted to the instability of a relativistic TD), which is necessary for solving a number of problems in astrophysics (see, for example, Ref. 127).

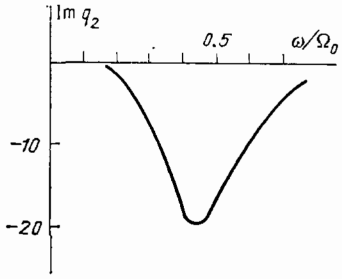

FIG. 19. Resonance amplification of sound by a vortex in an ideal liquid: Im$_{q_2}$ as a function of the frequency.

1) However the further evolution of vortices beneath the surface of a heavy liquid can in some cases be accompanied by emission of gravity waves.[128]

2) The use of an incorrect kinematic boundary condition in Ref. 63 led to the wrong result.

3) We point out here an analogy with drift waves in a nonuniform plasma, which under certain conditions can be described by the equations of the geostrophic approximation.[83] The results obtained below also hold, therefore, for drift waves on TD.

4) The weak amplificaion here is associated with the obvious fact that in a "narrow" shear layer there are few resonant particles.

5) We note in passing that taking into account the viscosity in a stratified medium does not eliminate the singularities in the corresponding generalized Orr-Sommerfeld equation, which was first derived by Drazin,[93] but merely lowers the order of the singularity. The singularity can be removed in this case by taking into account diffusion, which smears the stratification, but in so doing the order of the equation is increased to sixth order.[94]

1. R. Betchov and W. O. Criminale, Jr., of *Parallel Flow Stability* Academic Press, N. Y., 1967. [Russ. Transl., Mir, M., 1971].
2. A. Einstein, *Collection of Scientific Works* [Russ. trans.], Nauka, M., 1967, Vol. 4].
3. G. K. Batchelor, *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1967 [Russ. Trans. Mir, M., 1973].
4. Rayleigh (John William Strutt), *The Theory of Sound*, Dover Press, N.Y. 1945 [Russ. Transl., Gostekhizdat, M., 1955, vol. 2].
5. W. Heisenberg, Ann. Phys. Leipzig **74**, 577 (1924).
6. C. C. Kin, *The Theory of Hydrodynamic Stability*, Cambridge University Press, Cambridge, 1955 [Russ. Transl. IL, M., 1958].
7. W. Wasow *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, N.Y., 1966. [Russ. Transl., Mir, M., 1968].
8. M. A. Gol'dshtik and V. N. Shtern, *Hydrodynamic Instability and Turbulence* [in Russian], Nauka, Novosibirsk (1977).
9. K. M. Case, Phys. Fluids **3**, 366 (1960).
10. A. V. Timofeev, Usp. Fiz. Nauk **102**, 185 (1970) [Sov. Phys. Usp. **13**, 632 (1971)].
11. B. B. Kadomtsev and V. V. Kontorovich, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **17**, 511 (1974). [Radiophys. Quantum Electron. **17**, 386 (1974)].
12. N. S. Erokhin and S. S. Moiseev, Usp. Fiz. Nauk **109**, 225 (1973) [Sov. Phys. Usp. **16**, 64 (1973)].
13. M. I. Rabinovich and A. L. Fabrikant, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **19**, 721 (1976). [Radiophys. Quantum Electron. **19** (1976)].
14. A. V. Mikhaĭlovskiĭ, *Theory of Plasma Instabilities* Consultants Bureau, N.Y., 1974 [Russ. original Atomizdat, M., 1971].
15. B. B. Kadomtsev, *Collective Phenomena in Plasmas* [in Russian], Nauka, M., 1976.
16. M. I. Rabinovich and D. I. Trubetskov, *Introduction to the Theory of Oscillations and Waves* [in Russian], Nauka, M., 1974.
17. R. J. Briggs, J. D. Daugherty, and R. H. Levi, Phys. Fluids **13**, 421 (1970).
18. D. D. Ryutov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 446 (1975) [JETP Lett. **22**, 215 (1975)].
19. E. Meyer, 4th International Congress on Acoustics, Copenhagen, 1962, Congr. Rept., Copenhagen (1963), Vol. 2, p. 139.
20. A. A. Andronov and A. L. Fabrikant in *Nonlinear Waves* [in Russian], (Ed.) A. V. Gaponov, Nauka, M., 1979.
21. A. L. Fabrikant, Izv. Akad. Nauk SSSR, Ser. Fiz. Atmos. Okeana **12**, 858 (1976). [Izv. Atmos. Ocean. Phys. **12**, 524(1976)].
22. L. Sh. Tsimring, *ibid.* **19**, 68 (1983). [*ibid* **19**, 47 (1983)].
23. G. M. Golemshtok and A. L. Fabrikant, Akust. Zh. **26**, 383 (1980) [Sov. Phys. Acoust. **26**, 209 (1980)].
24. A. L. Fabrikant, *ibid* **28**, 694 (1982) [Sov. Phys. Acoust. **28**, 410 (1982)].
25. O. M. Phillips, *The Dynamics of the Top Layer of the Ocean* [Russ. Transl., Gidrometeoizdat, Leningrad, 1980].
26. E. Gossard and W. Hook, *Waves in the Atmosphere* [Russ. transl., Mir, M., 1978].
27. J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, N.Y., 1982 [Russ. transl. Mir, M., 1984], Vol. 1, 2.
28. M. E. Goldstein, *Aeroacoustics* [in Russian], Mashinostroenie, Moscow (1981).
29. G. A. Lupanov, Zh. Tekh. Fiz. **43**, 925 (1973) [Sov. Phys. Tech. Fiz. **18**, 587 (1973)].
30. V. D. Pikulin and N. S. Stepanov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **19**, 1564 (1976). [Radiophys. Quantum Electron. **19**, (1976)].
31. L. A. Ostrovskiĭ, S. A. Rybak, and L. Sh. Tsimring, Usp. Fiz. Nauk **150**, 417 (1986) [Sov. Phys. Usp. **29**, 1040 (1986)].
32. V. L. Ginzburg and V. A. Ugarov, Usp. Fiz. Nauk **118**, 175 (1976) [Sov. Phys. Usp. **19**, 94 (1976)].
33. M. MacIntyre, *Modern Hydrodynamics: Progress and Problems* [Russ. Transl., Mir, M., 1984, p. 154].
34. A. L. Fabrikant, "Momentum of a wave packet in a mdeium and induced flows" (In Russian), Preprint of the Institute of Applied Physics, Academy of Sciences of the USSR, Gor'kiĭ (1987).
35. A. B. Ezerskiĭ, L. A. Ostrovskiĭ, and Yu. A. Stepanyants, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **17**, 1201 (1981). [Izv. Atmos. Ocean. Phys. **17**, 890 (1981)].
36. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, Pergamon Press, N.Y., 1986. [Russ. original, Nauka, M., 1986].
37. J. D. Jackson, *Classical Electrodynamics*, John Wiley, N.Y., 1965 [Russ. transl. Mir, M., 1965].
38. A. B. Ezerskiĭ and V. V. Papko, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **22**, 979 (1986). [Izv. Atmos. Ocean. Phys. **22**, 756 (1986)].
39. S. R. Peierls, Proc. Roy. Soc. **347**, 475 (1975).
40. F. P. Bretherton, J. Fluid Mech. **36**, 795 (1969).
41. X. Zhu and J. R. Holton, J. Atmos. Sci. **44**, 620 (1987).
42. V. P. Reutov, Preprint No. 218, Institute of Applied Physics, Academy of Science, Gor'kiĭ (1988).
43. J. W. Miles, J. Acoust. Soc. Am. **29**, 226 (1957).
44. H. A. Ribner, *ibid.*, 435.
45. J. E. McKenzie, J. Geophys. Res. **77**, 2015 (1972).
46. G. A. Lupanov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **18**, 1711 (1975). [Radiofiz. Quantum Electron. **18**, 1263 (1975)].
47. J. W. Miles, J. Fluid Mech. **4**, 528 (1958).
48. S. I. Syrovatskiĭ, Zh. Eksp. Teor. Fiz. **27**, 121 (1954) not translated.
49. N. G. Kikina and D. G. Sannikov, Prikl. Mat. Mekh. No. 3, 580 (1969). [PMM J. Appl. Math. Mech. **33**, 567 (1969)].
50. M. V. Nezlin, Usp. Fiz. Nauk **150**, 1 (1986) [Sov. Phys. Usp. **29**, 807 (1986)].
51. P. I. Kolykhalov, Izv. Akad. Nauk SSSR, Ser. Mekh. zhidk. gaza, No. 3, 145 (1984). [Fluid Dyn. (1984)].
52. W. Blumen, P. G. Drazin, and D. F. Billings, J. Fluid Mech. **71**, 305 (1975).
53. E. B. Graham and P. B. Graham, J. Acoust. Soc. Am. **46**, 169 (1969).
54. V. V. Kurin, B. E. Nemtsov, and V. Ya. Éĭdman, Usp. Fiz. Nauk **147**, 157 (1985) [Sov. Phys. Usp. **28**, 827 (1985)].
55. D. S. Jones and J. B. Morgan, Proc. Cambr. Phil. Soc. **72**, 465 (1972).
56. M. A. Mironov, Akust. Zh. **21**, 79 (1975) [Sov. Phys. Acoust. **21**, 48 (1975)].
57. B. E. Nemtsov, Preprint No. 211, Scientific-Research Institute of Radiophysics, Gor'kiĭ (1986).
58. B. E. Nemtsov and L. Sh. Tsimring, Preprint No. 156, Institute of Applied Physics, Academy of Sciences of the USSR, Gor'kiĭ (1986).
59. I. P. Golyamin [Ed.], Ultrasound: Small Encyclopedia [in Russian], Sov. entsiklopediya, M., 1979.
60. A. A. Andronov and A. L. Fabrikant, Akust. Zh. **26**, 655 (1980) [Sov. Phys. Acoust. **26**, 370 (1980)].
61. V. I. Gaponov, Electronics [in Russian], Fizmatlit, M., 1960.
62. M. I. Petelin, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **28**, 354 (1985). [Radiofiz. Quantum Electron. **28**, (1985)].
63. J. W. Miles, J. Acoust. Soc. Am. **26**, 1015 (1954).
64. N. G. Kikina, Akust. Zh. **13**, 213 (1967) [Sov. Phys. Acoust. **13**, 284 (1967)].
65. R. A. Gerwin, Rev. Mod. Phys. **40**, 652 (1968).
66. D. V. Evans, J. Fluid Mech. **68**, 389 (1975).
67. A. Ya. Basovich, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **14**, 901 (1978). [Izv. Atmosph. Ocean. Phys. **14**, 637 (1978)].
68. L. A. Ostrovskiĭ, Yu. A. Stepanyants, and L. Sh. Tsimring in *Nonlinear Waves: Self-Organization* [in Russian], edited by A. V. Gaponov-Grekhov and M. I. Rabinovich, Nauka, M., 1983.
69. R. O. Cairns, J. Fluid Mech. **92**, 1 (1979).
70. M. V. Nezlin, Usp. Fiz. Nauk **120**, 481 (1976) [Sov. Phys. Usp. **19**, 946 (1976)].
71. L. A. Ostrovskiĭ and Yu. A. Stepanyants, Izv. Akad. Nauk SSSR, Ser. Mekhanika zhidkosti i gaza, No. 4, 63 (1982). [Fluid Dyn. (1982)].
72. T. P. Barnett and Kenyon, Rep. Prog. Phys. **38**, 667 (1975).
73. T. B. Benjamin, J. Fluid Mech. **16**, 436 (1963).
74. T. Kamatsu, M. Imasaka, and G. Tsubaki, Kidai nogaku syuko (Japanese) **52**, 25 (1979).
75. Yu. A. Stepanyants, Author's Abstract of Candidate's Dissertation in Physical-Mathematical Sciences, Gor'kiĭ (1983).
76. J. S. Turner, *Buoyancy Effects in Fluids*, Cambridge Univ. Press, 1973 [Russ. transl. Mir, M., 1977].
77. Yu. I. Troitskaya and A. L. Fabrikant, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **32**, 1521 (1989). [Radiophys. Quantum Electron. **32**, (1989)].
78. L. O. Goldstone and A. A. Oliner, IRE Trans. **AP-7**, 307 (1959).
79. L. A. Ostrovskiĭ and L. Sh. Tsimring, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **17**, 766 (1981). [Izv. Atmosph. Ocean. Phys. **17**, 564 (1981)].
80. N. S. Erokhin and S. S. Moiseev, Zh. Eksp. Teor. Fiz. **65**, 1431 (1973)

[Sov. Phys. JETP **38**, 712 (1974)].
[81]V. V. Zheleznyakov, V. V. Kocharovskiĭ, and Vl. V. Kocharovskiĭ, Zh. Eksp. Teor. Fiz. **87**, 1565 (1984) [Sov. Phys. JETP **60**, 897 (1984)].
[82]A. L. Fabrikant, Izv. Akad. Nauk SSSR, Ser. Fiz. Atmos. Okeana **23**, 106 (1987). [Izv. Atmosph. Ocean. Phys. **23**, 79 (1987)].
[83]A. Hasegawa and K. Mima, Phys. Fluids **21**, 87 (1978).
[84]L. D. Landau, *Collected Works* [in Russian], Nauka, M., 1969, Vol. 2.
[85]E. L. Aĭns, *Ordinary Differential Equations* [in Russian], GOTI, Khar-'kov (1933).
[86]L. A. Dikiĭ, Dokl. Akad. Nauk SSSR **135**, 1068 (1960) [Sov. Phys. Dokl. **5**, 1179 (1960)].
[87]K. Case, Hydrodynamic Instability [Russ. transl. Mir, M., 1960].
[88]A. L. Fabrikant, Akust. Zh. **22**, 107 (1976) [Sov. Phys. Acoust. **22**, 56 (1976)].
[89]V. G. Gavrilenko, L. A. Zelekson, and A. Ya. Basovich, *Abstracts of papers presented at the Symposium on the Physics of Acousto-Hydrodynamic Phenomena*, Sukhumi, 1975 (in Russian) p. 250.
[90]A. A. Andronov and A. L. Fabrikant, Akust. Zh. **26**, 655 (1980) [Sov. Phys. Acoust. **26**, 370 (1980)].
[91]J. W. Miles, J. Fluid Mech. **3**, 185 (1957).
[92]A. O. Vrobel' and V. M. Kontorovich, Pis'ma Astron Zh. **8**, 330 (1982). [Sov. Astron. Lett. **8**, 179 (1982)].
[93]G. Drazin, Proc. Cambridge Philos. Soc. **58**, 646 (1962).
[94]P. J. Hazel, J. Fluid Mech. **39**, 775 (1967).
[95]W. L. Jones, J. Fluid Mech. **34**, 609 (1968).
[96]I. A. Eltayeb and J. F. MeKenzie, *ibid.* **72**, 661 (1975).
[97]C. A. van Duin and H. Kelder, *ibid.* **120**, 505 (1982).
[98]R. S. Lindzen and J. W. Barker, *ibid.* **151**, 189 (1985).
[99]D. P. Lalas and F. Einaudi, J. Atmos. Sci. **33**, 1248 (1976).
[100]D. C. Fritts, *ibid.* **37**, 1642 (1980).
[101]A. Michalke and A. Timme, J. Fluid Mech. **39**, 647 (1961).
[102]V. F. Kop'ev and E. A. Leont'ev, Akust. Zh. **29**, 192 (1983) [Sov. Phys. Acoust. **39**, 111 (1983)].
[103]E. G. Broadbent and D. W. Moore, Philos. Trans. R. Soc. London A **390**, 353 (1979).
[104]Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. **62**, 2076 (1972) [Sov. Phys. JETP **35**, 1085 (1972)].
[105]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, N.Y., 1984 [Russ. original, Nauka, M., 1982].
[106]J. A. Fezer, Phys. Fluids **6**, 508 (1963).
[107]S. Duhan, F. Gratton, and G. Gratton, *ibid.* **14**, 2067 (1971).
[108]D. J. Acheson, J. Fluid Mech. **77**, 433 (1976).
[109]G. A. Lupanov, Izv. Vyssch. Uchebn. Zaved., Radiofiz. **18**, 1711 (1975). [Radiophys. Quantum. Electron. **18**, 1263 (1975)].
[110]V. G. Gavrilenko and L. A. Zelekson, *ibid.* **20**, 982 (1977). [*ibid.* **20** (1977)].
[111]V. G. Gavrilenko and L. A. Zelekson, Fiz. Plazmy **6**, 1046 (1980). [Sov. J. Plasma Phys.. **6**, 574 (1980)].
[112]D. J. Benney and R. F. Bergeron, Stud. Appl. Math. **48**, 181 (1969).
[113]R. Holerman, *ibid.* **51**, 139 (1972).
[114]V. P. Reutov, Zh. Prikl. Mekh. Tekh. Fiz., No. 4, 43 (1982). [J. Appl. Mech. Tech. Phys. (1982)].
[115]S. M. Churlov and I. G. Shukhman, J. Fluid Mech. **180**, 1 (1987).
[116]I. V. Alelekov and S. V. Shagalov, Abstracts of Reports at the 3rd Conference of Soviet Oceanographers. Section on the Physics and Chemistry of the Ocean: Wind, Internal, and Tidal waves; Tsunamis (in Russian), Gidrometeoizdat, Leningrad, 1987, p. 15.
[117]J. W. Coltman, J. Acoust. Soc. Am. **44**, 983 (1968).
[118]H. E. Willoughby, J. Atmos. Sci. **34**, 1028 (1977).
[119]R. V. Shchepetnov, V. A. Troitskaya, and B. V. Dovbnya, Dokl. Akad. Nauk SSSR **290**, 582 (1986) [Dokl. Earth Sciences (1986)].
[120]Collection of Papers on the Structure, Kinematics, and Dynamics of Synoptic Vortices [in Russian], Sevastopol' (1980).
[121]A. V. Gul'el'mi, *MHD Waves in the Circumterrestrial Plasma* [in Russian], Nauka, M., 1979.
[122]I. S. Veselovskiĭ, *Progress in Science and Technology. Series on the Investigation of Space,* [in Russian], VINITI. M., 1984, Vol. 22.
[123]S. A. Kaplan, S. B. Pikel'ner, and V. N. Tsytovich, *Physics of the Plasma in the Solar Atmosphere* [in Russian], Nauka, M., 1977.
[124]L. Kh. Ingel', Izv. Akad. Nauk SSSR, Ser. Mekhanika zhidkosti i gaza, No. 2, 144 (1983). [Fluid Dyn. (1983)].
[125]B. D. Turland and P. A. G. Schener, Mon. Not. R. Astron. Soc. **176**, 421 (1976).
[126]R. D. Blandford and J. E. Prinfe, *ibid.*, 652.
[127]M. C. Begelman, R. D. Blandford, and M. J. Rees, Rev. Mod. Phys. **56**, 255 (1984).
[128]M. V. Kurganskiĭ, Izv. Akad. Nauk SSSR, Ser. Mekhanika zhidkosti i gaza, No. 5, 184 (1986). [Fluid Dyn. (1986)].

Translated by M. E. Alferieff